

A LARGE DEFLECTION THEORY OF
ORTHOTROPIC SANDWICH SHALLOW SHELLS

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LIST OF SYMBOLS

$A, B,$	Constant
A^{273}, A^{373}	Elastic constant
$\vec{a}_\alpha, \vec{e}_\alpha,$	Tangential vector
$a_{\alpha/\beta}, e_{\alpha/\beta}, g_{ij},$	Metric tensor
$\vec{e}_3,$	Transverse unit vector
$D^{\alpha/\beta},$	Thermal coefficient
$E, E_r, E_\theta,$	Young's modulus
$G_1, G_2, G_3,$	Modulus of rigidity
$h,$	Total thickness of shell structure
$k,$	Index of shallowness of the shell structure
$M^{\alpha/\beta},$	Moment per unit length
$N^{\alpha/\beta},$	Force per unit length
$Q^{\alpha/\beta},$	Transverse shearing force per unit length
$q,$	Mechanical loading per unit area
$\vec{R}, \vec{r},$	Position vector
$r_0,$	Base radius
$R_0,$	Radius of curvature
$r, \vartheta, z,$	Cylindrical coordinates
$t,$	Thickness of the facings
$\bar{t},$	Thickness of core
$T,$	Temperature
$u_\alpha, \tilde{u}_\alpha,$	Tangential displacement

LIST OF SYMBOLS (Continued)

$\vec{V}$,	Displacement Vector
$w, \tilde{w}$,	Transverse deflection
w_0 ,	Non-dimensional central deflection
x_3 ,	Transverse coordinate
$\alpha_r, \alpha_\theta, \alpha$,	Thermal expansion coefficient
$\alpha_n, \bar{\alpha}_n, \beta_n, \bar{\beta}_n, \gamma_n, \bar{\gamma}_n$,	Coefficient in power series
δ ,	Variation symbol
$\epsilon_{ij}, \epsilon_{\alpha\beta}$,	Strain tensor
$\sigma^{\alpha\beta}$,	Stress tensor
θ_α ,	Angular displacement
ν_β ,	Unit normal to boundary
$\nu_{r\theta}, \nu_{\theta r}, \nu$,	Poisson's ratio

Abstract of Dissertation Presented to the Graduate Council
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A theoretical derivation of the equilibrium equations for a large deflection of sandwich shallow shells under mechanical as well as thermal loading has been carried out. The approach is in general coordinates so that it is inclusive enough to cover any specific reference frame for the actual engineering problem. By using the principle of minimum potential energy five coupled nonlinear differential equations in tensorial form are obtained.

In order to demonstrate the applicability of the above theory, a typical practical problem with axisymmetry is treated numerically. By using a perturbation method these nonlinear differential equations are transformed into several sets of linear differential equations. With the aid of power-series expansion technique the equations are solved successively. Finally, the problem is reduced

to a simpler case the numerical solution of which is available for comparison. The results turn out to be in good agreement with the known solutions.

INTRODUCTION

It has been known for a long time that sandwich construction is an efficient means of obtaining a light-weight structure with high strength. Modern developments in the aerospace industry have created an urgent need for such construction. Economical manufacturing methods are now being developed. As a result, more research work concerning these structures has become increasingly desirable.

For the sandwich shell the exact solution of the problem is very difficult to obtain because in this case the core, as well as the facings, has to be treated as a three-dimensional body. Numerous approximate theories in line with ordinary shell theory have been proposed in the past. Among these the first significant contribution to an understanding of the behavior of sandwich shells was made by E. Reissner (1), who assumed that the facings do not sustain any bending moment, i.e., the in-plane stresses are evenly distributed across the thickness of the facing, and the core is soft. The transverse compression and shear deformation of the core are considered. In the same paper, these effects, which differentiate the sandwich shell theory from the ordinary shell theory based on the Kirchhoff-Love

hypothesis, were evaluated for several types of sandwich structures. It was found that for certain geometrical configurations and material properties the effect of transverse normal stress deformation is negligibly small. Within this approach, Reissner also initiated a finite-deflection theory of sandwich plates (2, 3), based on the assumption that the core is in the small-deflection region and the facings act as membranes experiencing a finite deflection. The resultant equations permit the analysis of the effects of transverse shear deformation and the transverse normal stress deformation in the core on the over-all behavior of the sandwich plate. It was shown through an order-of-magnitude analysis that the effect of transverse normal stress in the core is negligibly small compared with the effect due to transverse shear stresses. It was further shown that the range of deflections for which the linear theory (small-deflection theory) is adequate decreases according to an explicit formula as the core is made softer relative to the facings. Since then, numerous papers concerning the structural properties of sandwich plates have been published. Some significant contributions (4, 5, 6, 7, 8, 9) have been summarized in (10). Wang (11) proposed a simplified method to derive the stress-displacement relation for the finite-deflection theory by using the principle of complementary energy. The result was applied

to set up the governing equations for cylindrical sandwich shells and also to study the stability of sandwich cylinders (12, 13). A rather complete survey of modern developments in the analysis of sandwich structures has been conducted by Habip (14). In the field of large-deflection theory of sandwich shallow shells, Karavanov (15) derived, by following Reissner's approach (2, 3), the nonlinear deflection equations taking account of the curvature effects. Grigoliuk has derived a set of governing differential equations for sandwich shells with a soft core (16), by considering that the facings transmit both the moments and tangential stresses in the finite-deflection region and the core transmits only the transverse shear in the small-deflection region. Later he also considered a case of finite deflection of sandwich shells with a stiff core, in which the membrane stresses were also taken into account (17). The same approach has been worked out by Wasicina (18) for the case of unsymmetrical construction. In 1961, Mush-tari (19) published a paper to discuss a general theory of shallow sandwich shells. In this paper, an unsymmetrical sandwich shell construction was proposed. The thermal effect was also taken into account. However, the temperature was limited to uniform distribution over the facings and linear distribution through the thickness of the core. In the same year, Fulton (20) presented a paper on a nonlinear

theory of shallow unsymmetrical sandwich shells of double curvature. Grigoliuk and Chulkov have presented a small-deflection theory of sandwich shells with stiff core (21). They also considered the compressibility in the transverse direction of the core. The core was taken as a three-dimensional body but with the assumption that the displacements could be approximately expressed as a linear function of the transverse coordinate. Both statical and dynamical cases were discussed. However, only the governing equations were set up and no solution to the equations was made available. Grigoliuk and Chulkov have published a paper on a general large-deflection theory of elastic sandwich shallow shells (22). In contrast to the paper given before, this paper dealt with an elastic shallow shell with different thicknesses of the facings. The facings were considered homogeneous and isotropic but of different materials. The core was assumed incompressible in the transverse direction, and the displacements in the tangential directions were assumed to be a linear function of the transverse coordinate. The reference surface was defined in terms of rectangular Cartesian coordinates. Thus far, the Russian works mentioned above do not include any solution to the governing equations.

This dissertation deals with a large-deflection theory of sandwich shallow shells under mechanical as well

as thermal loadings. The analysis is carried out in general coordinates. All the parameters are in terms of functions of the latter. Naturally, tensor notation is employed in the derivation. Through differential geometry the strains can be expressed in terms of the displacements, including the nonlinear terms, of the middle surface of a shell within an approximate theory of shallow shells. This relationship is applied to calculate the strain energies for the upper and lower facings as well as for the core which is considered as stiff and orthotropic. The total strain energy is obtained by summing those strain energies for the individual components of the sandwich structure. The variation principle of minimum potential energy is employed to obtain a set of nonlinear equations. Certain assumptions used in this derivation are: (a) the facings and core are thin, and (b) the Kirchhoff-Love hypothesis is valid for the facings while the tangential displacements for the core are linear functions of the thickness coordinate. The transverse deflection is assumed constant through the thickness of the whole structure.

Two methods are proposed in the derivation. In the first method, labeled method (A), the tangential displacements of the middle surfaces, u_{α}^I , u_{α}^{II} , of the upper and lower facings, respectively, and the transverse deflection w are taken as the independent parameters, as most

previous works have done. The second method, labeled method (B), takes $\bar{u}_\alpha$, the tangential displacements of the middle surface of the core, θ_α , the rotations of the cross-section normal to the middle surface of the core, and w , the transverse deflection of the structure, as the independent parameters. Both methods have specific merits in practical applications. Since there is no difference in principle, only method (B) is applied in a practical application in which all the tensorial equations are reduced to engineering form.

An axisymmetrical sandwich shallow shell of paraboloid geometry is introduced as an illustrative example to demonstrate the applicability of the proposed theory. A perturbation scheme using the dimensionless central deflection as a parameter is chosen to attack the nonlinear equilibrium equations obtained from the previous theoretical derivation. With this scheme the nonlinear differential equations are reduced to several sets of linear differential equations. In solving these linear differential equations certain series solutions with undefined coefficients are successively substituted into the differential equations. Together with the boundary conditions, a set of linear algebraic equations of the undefined coefficients are obtained. The matrix inversion is applied to solve for the undefined coefficients.

A digital computer program written in Fortran II provides the numerical results. Furthermore, the case is reduced to an ordinary shallow spherical shell to make some numerical comparisons with known solutions (23, 24, 25). This leads to good agreement.

CHAPTER I

THEORETICAL DERIVATION

1. Mathematical Preliminaries

The geometrical relations in plane coordinates for a shallow surface have been discussed by Green and Zerna (26). A net of coordinate curves on the middle surface M is considered to be projected on a plane π . The direction of projection is given by a unit vector $\vec{e}_3$ which is perpendicular to the plane π . Hence, there now is a one-to-one corresponding net of curvilinear coordinates between the π -plane and the surface M . The position vector $\vec{R}$ can be expressed as shown in Figure 1.

$$\vec{R} = \vec{r} + Z \vec{e}_3. \quad (1-1)$$

In this case the surface M can be represented by a net of curvilinear coordinates on the π -plane and a scalar quantity Z which is a function of the curvilinear coordinates on the π -plane.

The tangent vectors to the curves on the surface M can be expressed by

$$\vec{a}_\alpha = \vec{R}_{,\alpha} = \vec{r}_{,\alpha} + Z_{,\alpha} \vec{e}_3 = \vec{e}_\alpha + Z_{,\alpha} \vec{e}_3, \quad (1-2)$$

where $\vec{e}_\alpha = \vec{r}_{,\alpha}$ and the comma means partial derivative with respect to the indicated variable.

The displacement vector can also be expressed by

$$\vec{U} = \tilde{u}_\alpha \vec{a}^\alpha + \tilde{w} \vec{a}_3 = u_\alpha \vec{e}^\alpha + w \vec{e}_3, \quad \alpha = 1, 2, \quad (1-3)$$

where $\vec{a}^i$, $i = 1, 2, 3$, and $\vec{e}^i$, $i = 1, 2, 3$, are the right hand triads on the surface M and the π -plane, respectively, and $\tilde{u}_\alpha$, $\tilde{w}$ and u_α , w are the displacement components with respect to the base vectors $\vec{a}^i$ and $\vec{e}^i$, respectively. For a shallow shell it is understood that the values of Z and its derivatives up to the second order are small in comparison to the other dimensions of the shell. Then, as an approximation, we neglect terms containing the squares and products of Z as well as its derivatives up to the second order with the other quantities. Thus,

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta = (\vec{e}_\alpha + Z_{,\alpha} \vec{e}_3) \cdot (\vec{e}_\beta + Z_{,\beta} \vec{e}_3) \doteq e_{\alpha\beta}. \quad (1-4)$$

Similarly,

$$a^{\alpha\beta} \doteq e^{\alpha\beta}, \quad (1-5)$$

and

$$a = |a_{\alpha\beta}| \doteq |e_{\alpha\beta}| = e. \quad (1-6)$$

Multiplying both sides of Equation (1-2) by $e^{\alpha\beta}$ and applying Equation (1-4), we can obtain

$$\vec{a}^\beta = \vec{e}^\beta + e^{\alpha\beta} Z_{,\alpha} \vec{e}_3. \quad (1-7)$$

From vector algebra we have

$$\vec{a}_\alpha \times \vec{a}_\beta = \epsilon_{\alpha\beta} \vec{a}_3 = \pm \sqrt{e} \vec{a}_3, \quad \alpha \neq \beta \quad (1-8a)$$

$$\vec{a}_3 \times \vec{a}_\beta = \epsilon_{\beta\alpha} \vec{a}^\alpha. \quad (1-8b)$$

Substituting Equation (1-2) into Equation (1-8), and assigning $\alpha = 1, \beta = 2$, we have

$$\sqrt{e} \vec{a}_3 = (\vec{e}_1 + Z_{,1} \vec{e}_3) \times (\vec{e}_2 + Z_{,2} \vec{e}_3) = \sqrt{e} (\vec{e}_3 - Z_{,\alpha} \vec{e}^\alpha). \quad (1-9)$$

Therefore,

$$\vec{a}_3 = \vec{e}_3 - Z_{,\alpha} \vec{e}^\alpha, \quad (1-10)$$

or

$$\vec{e}_3 = \vec{a}_3 + Z_{,\alpha} \vec{e}^\alpha. \quad (1-11)$$

Using Equation (1-7), Equation (1-11) becomes

$$\vec{e}_3 = \vec{a}_3 + Z_{,\alpha} (\vec{a}^\alpha - e^{\alpha\beta} Z_{,\beta} \vec{e}_3). \quad (1-12a)$$

Neglecting terms involving second-order products of Z-terms, we obtain

$$\vec{e}_j = \vec{a}_j + Z_{,\alpha} \vec{a}^\alpha. \quad (1-12b)$$

In order to calculate the strain components in terms of the displacement vector we define the displacement vector (see Figure 2) as

$$\vec{V} = \vec{R} - \vec{r} + \vec{c}. \quad (1-13)$$

The Lagrangian strain tensor is defined as (27)

$$E_{ij} = \frac{1}{2}(G_{ij} - g_{ij}). \quad (1-14)$$

Where G_{ij} and g_{ij} are metric tensors of the initial and deformed body, respectively. Performing the partial differentiation in Equation (1-13) we obtain

$$\vec{G}_i = \vec{R}_{,i} = \vec{V}_{,i} + \vec{g}_i. \quad (1-15)$$

Also, by definition

$$\begin{aligned} G_{ij} &= \vec{G}_i \cdot \vec{G}_j = (\vec{g}_i + \vec{V}_{,i}) \cdot (\vec{g}_j + \vec{V}_{,j}) \\ &= g_{ij} + \vec{g}_j \cdot \vec{V}_{,i} + \vec{g}_i \cdot \vec{V}_{,j} + \vec{V}_{,i} \cdot \vec{V}_{,j}. \end{aligned} \quad (1-16)$$

Substituting Equation (1-16) into Equation (1-14), we obtain the Lagrangian strain tensor

$$\mathcal{E}_{ij} = \frac{1}{2}(\vec{g}_i \cdot \vec{V}_j + \vec{g}_j \cdot \vec{V}_i + \vec{V}_i \cdot \vec{V}_j), \quad (1-17a)$$

and on a surface the strain tensor can be expressed by

$$\mathcal{E}_{\alpha\beta} = \frac{1}{2}(\vec{a}_\alpha \cdot \vec{V}_\beta + \vec{a}_\beta \cdot \vec{V}_\alpha + \vec{V}_\alpha \cdot \vec{V}_\beta). \quad (1-17b)$$

Referring to the surface M, we can also write the displacement vector in the form

$$\vec{V} = \tilde{u}_\alpha \vec{a}^\alpha + \tilde{w} \vec{a}_3, \quad (1-18a)$$

and referring to the π -plane, as

$$\vec{V} = u_\alpha \vec{e}^\alpha + w \vec{e}_3. \quad (1-18b)$$

Hence

$$\tilde{u}_\alpha \vec{a}^\alpha + \tilde{w} \vec{a}_3 = u_\alpha \vec{e}^\alpha + w \vec{e}_3. \quad (1-19a)$$

Using Equation (1-7) and Equation (1-11), we find

$$\tilde{u}_\alpha \vec{a}^\alpha + \tilde{w} \vec{a}_3 = (u_\alpha + w z_{,\alpha}) \vec{a}^\alpha + (w - u_\alpha \vec{e}^\beta z_{,\beta}) \vec{a}_3. \quad (1-19b)$$

Hence

$$\tilde{u}_\alpha = u_\alpha + w z_{,\alpha}, \quad (1-20a)$$

$$\tilde{W} = W - U_{\alpha} e^{\alpha\beta} Z_{,\beta} . \quad (1-20b)$$

Also

$$\vec{V}_{,\alpha} = U_{\gamma/\alpha} \vec{e}^{\gamma} + W_{,\alpha} \vec{e}_3 , \quad (1-21)$$

where (/) means covariant derivatives with respect to the indicated variable. Substituting Equations (1-2) and (1-21) into Equation (1-17b), we obtain the strain tensor

$$\tilde{\epsilon}_{\alpha\beta} = \frac{1}{2} (U_{\alpha/\beta} + Z_{,\alpha} W_{,\beta} + U_{\beta/\alpha} + Z_{,\beta} W_{,\alpha} + U_{\gamma/\alpha} U_{\delta/\beta} e^{\gamma\delta} + W_{,\alpha} W_{,\beta}) . \quad (1-22a)$$

Since the displacements in the middle surface are still small even though we consider a large-deflection case, by neglecting the higher order terms $U_{\gamma/\alpha} U_{\delta/\beta} e^{\gamma\delta}$ in comparison with the other terms, we can express the strain tensor in terms of the displacements

$$\epsilon_{\alpha\beta} = \frac{1}{2} (U_{\alpha/\beta} + U_{\beta/\alpha} + Z_{,\alpha} W_{,\beta} + Z_{,\beta} W_{,\alpha} + W_{,\alpha} W_{,\beta}) . \quad (1-22b)$$

Corresponding strain-displacement relations have been given to rectangular Cartesian coordinates by Marguerre (28), and in cylindrical coordinates by Nash (23). Here an alternate derivation has been presented.

2. Derivation of Equations

A. Method A

The displacement of the middle surface of the upper facing, u_{α}^I , and the displacement of the middle surface of

the lower facing, u''_α , and the transverse deflection w are taken as the reference variables.

a. Strain-Displacement Relations and Strain Energy of a Sandwich Structure (Figure 3)

The facings are considered thin and the Kirchhoff-Love hypothesis is valid. The displacements for the facings can be expressed by

$$u_\alpha = u_\alpha^0 - (x_3 - x_3^0) W_{,\alpha} \quad , \quad (1-23)$$

where u_α^0 are the displacements at the middle surface and x_3^0 are the coordinates of the middle surfaces of the facings. For the upper facing, with $u_\alpha^0 = u'_\alpha$, $x_3^0 = \frac{1}{2}(t + \bar{t})$, and $\frac{1}{2}\bar{t} \leq x_3 \leq t + \frac{\bar{t}}{2}$, the displacements can be written as

$$u_\alpha = u'_\alpha - \left[x_3 - \frac{1}{2}(t + \bar{t}) \right] W_{,\alpha} \quad . \quad (1-24)$$

The strains can be obtained by substituting Equation (1-24) into Equation (1-22b)

$$\epsilon_{\alpha\beta} = \epsilon'_{\alpha\beta} - \left[x_3 - \frac{1}{2}(t + \bar{t}) \right] W_{,\alpha\beta} \quad , \quad (1-25)$$

where

$$\epsilon'_{\alpha\beta} = \frac{1}{2} (u'_{\alpha/\beta} + u'_{\beta/\alpha} + Z_{,\alpha} W_{,\beta} + Z_{,\beta} W_{,\alpha} + W_{,\alpha} W_{,\beta}) \quad . \quad (1-26)$$

For the lower facing, with $u_\alpha^0 = u''_\alpha$, $x_3^0 = \frac{1}{2}(t + \bar{t})$, and

$-t - \frac{\bar{t}}{2} \leq x_3 \leq -\frac{\bar{t}}{2}$, the displacements can be written as

$$u_\alpha = u_\alpha'' - \left[x_3 + \frac{1}{2}(t + \bar{t}) \right] w_{,\alpha} \quad (1-27)$$

The strains can be obtained by substituting Equation (1-27) into Equation (1-22b)

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}'' - \left[x_3 + \frac{1}{2}(t + \bar{t}) \right] w_{,\alpha\beta} \quad (1-28)$$

where

$$\varepsilon_{\alpha\beta}'' = \frac{1}{2} (u_{\alpha/\beta}'' + u_{\beta/\alpha}'' + z_{,\alpha} w_{,\beta} + z_{,\beta} w_{,\alpha} + w_{,\alpha} w_{,\beta}) \quad (1-29)$$

The variation of strain energy for the upper facing is

$$\delta U' = \int_V \sigma^{\alpha\beta} \delta \varepsilon_{\alpha\beta} dV = \int_V \sigma^{\alpha\beta} \left[\delta \varepsilon_{\alpha\beta}' - \left(x_3 - \frac{t + \bar{t}}{2} \right) \delta w_{,\alpha\beta} \right] dV \quad (1-30)$$

Where $\sigma^{\alpha\beta}$ is the stress tensor for the upper facing.

The variation of strain energy for the lower facing is

$$\delta U'' = \int_V \sigma''^{\alpha\beta} \delta \varepsilon_{\alpha\beta} dV = \int_V \sigma''^{\alpha\beta} \left[\delta \varepsilon_{\alpha\beta}'' - \left(x_3 + \frac{t + \bar{t}}{2} \right) \delta w_{,\alpha\beta} \right] dV \quad (1-31)$$

where $\sigma''^{\alpha\beta}$ is the stress tensor for the lower facing.

For the core, $-\frac{\bar{t}}{2} \leq x_3 \leq \frac{\bar{t}}{2}$, and since we consider a stiff orthotropic core, we have to consider the strain energy due to the deformation of the middle surface and the transverse shear deformation. Under this circumstance, the Kirchhoff-Love hypothesis no longer holds for the core. However, it is valid to assume that planes normal to the

middle surface of the core remain plane after deformation but no longer necessarily normal to the middle surface because of transverse shear deformation in the core. Under this assumption, the displacements in the core are linear functions of the distance measured from the middle surface of the core. We let $u_\alpha = A + Bx_3$, where A and B are constants to be determined by applying the geometrical continuity at the interfaces. At the interface between the upper facing and the core, i.e., at $x_3 = \frac{\bar{t}}{2}$, the displacements are given by Equation (1-24). Thus

$$A + B \frac{\bar{t}}{2} = u'_\alpha + \frac{t}{2} W_{/\alpha} . \quad (1-32a)$$

At the interface between the core and the lower facing, i.e., $x_3 = -\frac{\bar{t}}{2}$, from Equation (1-27) we have

$$A - B \frac{\bar{t}}{2} = u''_\alpha - \frac{t}{2} W_{/\alpha} . \quad (1-32b)$$

Solving Equations (1-32a) and (1-32b) for A and B , we obtain

$$A = \frac{1}{2} (u'_\alpha + u''_\alpha) , \quad (1-33a)$$

$$B = \frac{1}{\bar{t}} (u'_\alpha - u''_\alpha + t W_{/\alpha}) . \quad (1-33b)$$

Hence, the displacements in the core can be expressed as

$$u_\alpha = \frac{1}{2} (u'_\alpha + u''_\alpha) + \frac{\lambda_3}{\bar{t}} (u'_\alpha - u''_\alpha + t W_{/\alpha}) . \quad (1-34)$$

The strains can be obtained by substituting Equation (1-34) into Equation (1-22b) so that

$$\begin{aligned} \mathcal{E}_{\alpha\beta} = & \frac{1}{2} \left[\frac{1}{2} (u'_{\alpha/\beta} + u''_{\alpha/\beta}) + \frac{\gamma_3}{\epsilon} (u'_{\alpha/\beta} - u''_{\alpha/\beta} + t w_{\alpha/\beta}) \right. \\ & + \frac{1}{2} (u'_{\beta/\alpha} + u''_{\beta/\alpha}) + \frac{\gamma_3}{\epsilon} (u'_{\beta/\alpha} - u''_{\beta/\alpha} + t w_{\beta/\alpha}) \\ & \left. + w_{\alpha} w_{\beta} + \epsilon_{\alpha} w_{\beta} + \epsilon_{\beta} w_{\alpha} \right]. \end{aligned} \quad (1-35)$$

The transverse shear strains can be obtained from Equation (1-17) as follows

$$\mathcal{E}_{\alpha 3} = \frac{1}{2} (\vec{a}_{\alpha} \cdot \vec{V}_{,3} + \vec{a}_3 \cdot \vec{V}_{,\alpha} + \vec{V}_{,\alpha} \cdot \vec{V}_3). \quad (1-36)$$

Since $\vec{V} = u_{\alpha} \vec{a}^{\alpha} + w \vec{a}_3$, we have $\vec{V}_{,3} = u_{\alpha/3} \vec{a}^{\alpha} + w_{,3} \vec{a}_3$; the last term equals zero since w is assumed to be independent of x_3 . So that we have

$$\vec{V}_{,3} = u_{\alpha/3} \vec{a}^{\alpha}. \quad (1-37)$$

Substituting Equations (1-21) and (1-37) into Equation (1-36), we obtain the transverse shear strains in terms of the displacements

$$\mathcal{E}_{\alpha 3} = \frac{1}{2} (u_{\alpha/3} + w_{,\alpha} + u_{\gamma/\alpha} u_{\gamma 3} a^{\gamma}). \quad (1-38)$$

By the same argument utilized before, the higher order terms $u_{\gamma/\alpha} u_{\gamma 3} a^{\gamma}$ are neglected. Then,

$$\mathcal{E}_{\alpha\beta} = \frac{1}{2} (u_{\alpha/\beta} + w_{,\alpha}) . \quad (1-39)$$

When Equation (1-34) is used, Equation (1-39) becomes

$$\mathcal{E}_{\alpha\beta} = \frac{1}{2\bar{E}} [u'_{\alpha} - u''_{\alpha} + (t + \bar{E}) w_{,\alpha}] . \quad (1-40)$$

The variation of strain energy for the core is

$$\begin{aligned} \delta \bar{U} &= \int_V \bar{\sigma}^{\alpha\beta} \delta \mathcal{E}_{\alpha\beta} dV + 2 \int_V \bar{\sigma}^{\alpha 3} \delta \mathcal{E}_{\alpha 3} dV \\ &= \int_V \bar{\sigma}^{\alpha\beta} \delta \left[\frac{1}{2} (u'_{\alpha/\beta} + u''_{\alpha/\beta}) + \frac{\lambda_2}{\bar{E}} (u'_{\alpha/\beta} - u''_{\alpha/\beta} \right. \\ &\quad \left. + t w_{,\beta}) + \bar{z}_{\alpha} w_{,\beta} + w_{,\alpha} w_{,\beta} \right] dV + \int_V \bar{\sigma}^{\alpha 3} \cdot \\ &\quad \frac{1}{\bar{E}} [u'_{\alpha} - u''_{\alpha} + (t + \bar{E}) w_{,\alpha}] dV , \end{aligned} \quad (1-41)$$

where $\bar{\sigma}^{\alpha\beta}$ are the normal stresses and $\bar{\sigma}^{\alpha 3}$ are the transverse shearing stresses in the core.

b. Derivation of the Equilibrium Equations

The theorem of minimum potential energy (29) is applied to derive a system of equilibrium equations.

The total potential energy for the sandwich shallow shell is

$$P.E. = U' + U'' + \bar{U} - \int_A \bar{f} w dA . \quad (1-42)$$

The theorem of minimum potential energy states

$$\delta(P.E.) = \delta U' + \delta U'' + \delta U - \int_V g \delta W dA = 0. \quad (1-43)$$

From Equation (1-30)

$$\begin{aligned} \delta U' &= \int_V \sigma_{, \alpha \beta} [\delta u'_{, \alpha \beta} + w_{, \alpha} \delta w_{, \beta} + z_{, \alpha} \delta w_{, \beta} - (x_{, \beta} - z_{, \beta} \frac{\partial}{\partial z}) \delta w_{, \alpha \beta}] dV \\ &= \int_V [N_{, \alpha \beta} (\delta u'_{, \alpha \beta} + w_{, \alpha} \delta w_{, \beta} + z_{, \alpha} \delta w_{, \beta}) - M_{, \alpha \beta} \delta w_{, \alpha \beta}] dA, \end{aligned} \quad (1-44)$$

where

$$N_{, \alpha \beta} = \int_{-\frac{z}{2}}^{\frac{z}{2}} \sigma_{, \alpha \beta} d\alpha, \quad (1-45a)$$

$$M_{, \alpha \beta} = \int_{-\frac{z}{2}}^{\frac{z}{2}} \sigma_{, \alpha \beta} (x_{, \beta} - z_{, \beta} \frac{\partial}{\partial z}) d\alpha. \quad (1-45b)$$

When the divergence theorem is applied, Equation (1-44)

becomes

$$\begin{aligned} \delta U' &= \int_V \{ \sigma_{, \alpha \beta} [\delta u'_{, \alpha \beta} + w_{, \alpha} \delta w_{, \beta} + z_{, \alpha} \delta w_{, \beta}] + M_{, \alpha \beta} \delta w_{, \alpha \beta} \\ &+ \sigma_{, \alpha \beta} \delta w_{, \alpha \beta} N_{, \alpha \beta} \} dV + \int_V \{ M_{, \alpha \beta} \delta u'_{, \alpha \beta} + M_{, \alpha \beta} \delta w_{, \alpha \beta} N_{, \alpha \beta} \\ &+ \sigma_{, \alpha \beta} \delta w_{, \alpha \beta} (x_{, \beta} - z_{, \beta} \frac{\partial}{\partial z}) + M_{, \alpha \beta} \delta w_{, \alpha \beta} \} dV. \end{aligned} \quad (1-46)$$

Similarly from Equation (1-31)

$$\begin{aligned} \delta U'' = & - \int_A \left\{ N''^{\alpha\beta} \delta u''_\alpha + [N''^{\alpha\beta} (w_\alpha + z_\alpha)]_{,\beta} \delta w \right. \\ & + M''^{\alpha\beta} \delta w \} dA + \oint_S \left\{ N''^{\alpha\beta} \delta u''_\alpha \right. \\ & + [N''^{\alpha\beta} (w_\alpha + z_\alpha) + M''^{\alpha\beta}] \delta w \} \nu_\beta dS \\ & - \oint_S M''^{\alpha\beta} \delta w_{,\alpha} \nu_\beta dS, \end{aligned} \quad (1-47)$$

where

$$N''^{\alpha\beta} = \int_{-\frac{\bar{t}}{2}-t}^{-\frac{\bar{t}}{2}} \sigma''^{\alpha\beta} d\chi_3, \quad (1-48a)$$

$$M''^{\alpha\beta} = \int_{-\frac{\bar{t}}{2}-t}^{-\frac{\bar{t}}{2}} \sigma''^{\alpha\beta} (\chi_3 + \frac{t+\bar{t}}{2}) d\chi_3, \quad (1-48b)$$

and from Equation (1-41), we have

$$\begin{aligned} \delta \bar{U} = & - \int_A \left\{ \bar{N}^{\alpha\beta} \frac{1}{2} (\delta u'_\alpha + \delta u''_\alpha) + [\bar{N}^{\alpha\beta} (z_\alpha \right. \\ & + w_\alpha)]_{,\beta} \delta w + \frac{1}{\bar{\epsilon}} \bar{M}^{\alpha\beta} (\delta u'_\alpha - \delta u''_\alpha) - \frac{t}{\bar{\epsilon}} \bar{M}^{\alpha\beta} \delta w \\ & - \frac{Q^{\alpha\beta}}{\bar{\epsilon}} (\delta u'_\alpha - \delta u''_\alpha) + \frac{t+\bar{t}}{\bar{\epsilon}} Q^{\alpha\beta} \delta w \} dA \\ & + \oint_S \left\{ \bar{N}^{\alpha\beta} \left[\frac{1}{2} (\delta u'_\alpha + \delta u''_\alpha) + (z_\alpha + w_\alpha) \delta w \right] + \frac{1}{\bar{\epsilon}} \bar{M}^{\alpha\beta} \right. \\ & \left. (\delta u'_\alpha - \delta u''_\alpha) - \frac{t}{\bar{\epsilon}} \bar{M}^{\alpha\beta} \delta w + \frac{t}{\bar{\epsilon}} \bar{M}^{\alpha\beta} \delta w_{,\alpha} + \frac{t+\bar{t}}{\bar{\epsilon}} Q^{\beta\gamma} \delta w \right\} \nu_\beta dS, \end{aligned} \quad (1-49)$$

where

$$\bar{N}^{\alpha\beta} = \int_{-\frac{\bar{L}}{2}}^{\frac{\bar{L}}{2}} \bar{\sigma}^{\alpha\beta} d\chi_3, \quad (1-50a)$$

$$\bar{M}^{\alpha\beta} = \int_{-\frac{\bar{L}}{2}}^{\frac{\bar{L}}{2}} \bar{\sigma}^{\alpha\beta} \chi_3 d\chi_3, \quad (1-50b)$$

$$Q^{\alpha 3} = \int_{-\frac{\bar{L}}{2}}^{\frac{\bar{L}}{2}} \bar{\sigma}^{\alpha 3} d\chi_3. \quad (1-50c)$$

Substituting Equations (1-46), (1-47) and (1-49) into Equation (1-43), we obtain

$$\begin{aligned} & - \int_A \left\{ (N'^{\alpha\beta} + \frac{1}{2} \bar{N}^{\alpha\beta} + \frac{1}{\bar{E}} \bar{M}^{\alpha\beta} - \frac{1}{\bar{E}} Q^{\alpha 3}) \delta u'_\alpha \right. \\ & + (N''^{\alpha\beta} + \frac{1}{2} \bar{N}^{\alpha\beta} - \frac{1}{\bar{E}} \bar{M}^{\alpha\beta} + \frac{1}{\bar{E}} Q^{\alpha 3}) \delta u''_\alpha \\ & + \left\{ [(N'^{\alpha\beta} + N''^{\alpha\beta} + \bar{N}^{\alpha\beta})(W_\alpha + Z_\alpha)]_{,\beta} + (M'^{\alpha\beta}_{,\beta} \right. \\ & + M''^{\alpha\beta}_{,\beta} - \frac{t}{\bar{E}} \bar{M}^{\alpha\beta}_{,\beta} + \frac{t+\bar{t}}{\bar{E}} Q^{\alpha 3}_{,\alpha} + g) \left. \right\} \delta w \left. \right\} dA \\ & + \oint_S \left\{ (N'^{\alpha\beta} + \frac{1}{2} \bar{N}^{\alpha\beta} + \frac{1}{\bar{E}} \bar{M}^{\alpha\beta}) \delta u'_\alpha + (N''^{\alpha\beta} + \bar{N}^{\alpha\beta} \right. \\ & - \frac{1}{\bar{E}} \bar{M}^{\alpha\beta}) \delta u''_\alpha + [(N'^{\alpha\beta} + N''^{\alpha\beta} + \bar{N}^{\alpha\beta})(Z_\alpha + W_\alpha) + (M'^{\alpha\beta}_{,\beta} \\ & + M''^{\alpha\beta}_{,\beta} - \frac{t}{\bar{E}} \bar{M}^{\alpha\beta}_{,\beta} + \frac{t+\bar{t}}{\bar{E}} Q^{\alpha 3}_{,\alpha})] \delta w - (M'^{\alpha\beta} + M''^{\alpha\beta} \\ & + \frac{t}{\bar{E}} \bar{M}^{\alpha\beta}) \delta w_\alpha \left. \right\} n_\beta dS = 0. \quad (1-51) \end{aligned}$$

Since δu_α^0 , $\delta u_\alpha''$ and δw are independent variations the coefficients of these terms in the surface integral should vanish in order to satisfy Equation (1-51). Hence, we have the five simultaneous equilibrium equations

$$(N'^{\alpha\beta} + \frac{1}{2} \bar{N}^{\alpha\beta})_{/\beta} + \frac{1}{\hat{t}} (\bar{M}^{\alpha\beta}_{/\beta} - Q^{\alpha 3}) = 0, \quad (1-52a)$$

$$(N''^{\alpha\beta} + \frac{1}{2} \bar{N}^{\alpha\beta})_{/\beta} - \frac{1}{\hat{t}} (\bar{M}^{\alpha\beta}_{/\beta} - Q^{\alpha 3}) = 0, \quad (1-52b)$$

$$[N^{\alpha\beta}(w_{/\alpha} + z_{/\alpha})]_{/\beta} + (M^{\alpha\beta} - \hat{t} \bar{M}^{\alpha\beta})_{/\alpha\beta} + \hat{t} Q^{\alpha 3}_{/\alpha} + \hat{f} = 0, \quad (1-52c)$$

where

$$N^{\alpha\beta} = N'^{\alpha\beta} + N''^{\alpha\beta} + \bar{N}^{\alpha\beta}, \quad (1-53a)$$

$$M^{\alpha\beta} = M'^{\alpha\beta} + M''^{\alpha\beta} + \bar{M}^{\alpha\beta}, \quad (1-53b)$$

$$\hat{t} = \frac{t + \bar{t}}{\bar{t}}. \quad (1-53c)$$

and the boundary conditions

$$(N'^{\alpha\beta} + \frac{1}{2} \bar{N}^{\alpha\beta} + \frac{1}{\hat{t}} \bar{M}^{\alpha\beta}) \delta u'_\alpha \nu_\beta = 0, \quad (1-54a)$$

$$(N''^{\alpha\beta} + \frac{1}{2} \bar{N}^{\alpha\beta} - \frac{1}{\bar{E}} \bar{M}^{\alpha\beta}) \delta u''_{\alpha} \gamma_{\beta} = 0, \quad (1-54b)$$

$$[N^{\alpha\beta}(W_{\alpha} + z_{\alpha}) + (M^{\alpha\beta}_{\alpha} - \hat{t} M^{\alpha\beta}_{\alpha} + \hat{t} Q^{\beta 3})] \gamma_{\beta} \delta w = 0, \quad (1-54c)$$

$$(M'^{\alpha\beta} + M''^{\alpha\beta} + \frac{\hat{t}}{\bar{E}} \bar{M}^{\alpha\beta}) \delta W_{\alpha} \gamma_{\beta} = 0. \quad (1-54d)$$

So far, we have obtained a set of equilibrium equations and boundary conditions in terms of the stress and moment resultants. Now we want to express these equations and boundary conditions in terms of the displacements. Let us use a generalized form of the Duhamel-Neumann equations for an orthotropic material under the generalized plane-stress assumption

$$\sigma^{\alpha\beta} = A^{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta} + D^{\alpha\beta} T, \quad (1-55a)$$

$$\sigma^{\alpha 3} = A^{\alpha 3\gamma 3} \epsilon_{\gamma 3}, \quad (1-55b)$$

where $A^{\alpha\beta\gamma\delta}$ and $A^{\alpha 3\gamma 3}$ are the elastic coefficients which depend on the metric tensor as well as the properties of the undeformed body. T is the temperature function. $D^{\alpha\beta}$ are thermal coefficients. Substituting Equations (1-25), (1-26) and (1-55) into Equations (1-45a) and (1-45b),

and using the symmetric properties of the coefficients, we have

$$\begin{aligned}
 N'^{\alpha\beta} &= \int_{\frac{\bar{t}}{2}}^{t+\frac{\bar{t}}{2}} \left\{ A^{\alpha\beta\gamma\delta} \left[(U'_{\gamma\delta} + Z_{\gamma} W_{\delta} + \frac{1}{2} W_{\gamma} W_{\delta}) \right. \right. \\
 &\quad \left. \left. - (\chi_3 - \frac{t+\bar{t}}{2}) W_{\gamma\delta} \right] + D^{\alpha\beta} T' \right\} d\chi_3 \\
 &= A^{\alpha\beta\gamma\delta} \left[t (U'_{\gamma\delta} + Z_{\gamma} W_{\delta} + \frac{1}{2} W_{\gamma} W_{\delta}) \right] + T'^{\alpha\beta},
 \end{aligned} \tag{1-56a}$$

where

$$T'^{\alpha\beta} = \int_{\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2}+t} D^{\alpha\beta} T' d\chi_3 \tag{1-56b}$$

$$\begin{aligned}
 M'^{\alpha\beta} &= \int_{\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2}+t} \left\{ A^{\alpha\beta\gamma\delta} \left[(U'_{\gamma\delta} + Z_{\gamma} W_{\delta} + \frac{1}{2} W_{\gamma} W_{\delta}) \right. \right. \\
 &\quad \left. \left. - (\chi_3 - \frac{t+\bar{t}}{2}) W_{\gamma\delta} + D^{\alpha\beta} T' \right] (\chi_3 - \frac{t+\bar{t}}{2}) \right\} d\chi_3 \\
 &= -A^{\alpha\beta\gamma\delta} \frac{t^3}{12} W_{\gamma\delta} + M'_T{}^{\alpha\beta},
 \end{aligned} \tag{1-56c}$$

$$M'_T{}^{\alpha\beta} = \int_{\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2}+t} D^{\alpha\beta} T' (\chi_3 - \frac{t+\bar{t}}{2}) d\chi_3. \tag{1-56d}$$

Similarly, by Equations (1-48a) and (1-48b), we have

$$N''^{\alpha\beta} = A^{\alpha\beta\gamma\delta} t (U''_{\gamma\delta} + Z_{\gamma} W_{\delta} + \frac{1}{2} W_{\gamma} W_{\delta}) + T''^{\alpha\beta}, \tag{1-57a}$$

where

$$T''^{\alpha\beta} = \int_{-\frac{\bar{t}}{2}-t}^{-\frac{\bar{t}}{2}} D^{\alpha\beta} T'' d\chi_3, \quad (1-57b)$$

$$M''^{\alpha\beta} = -A^{\alpha\beta\gamma\delta} \frac{t^3}{12} W_{\gamma\delta} + M_T''^{\alpha\beta}, \quad (1-57c)$$

$$M_T''^{\alpha\beta} = \int_{-\frac{\bar{t}}{2}-t}^{-\frac{\bar{t}}{2}} D^{\alpha\beta} T''(\chi_3 + \frac{t+\bar{t}}{2}) d\chi_3. \quad (1-57d)$$

Also, applying Equations (1-55a), (1-55b) and (1-40) to Equations (1-50a), (1-50b) and (1-50c), we obtain

$$\begin{aligned} \bar{N}^{\alpha\beta} &= \int_{-\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2}} \left\{ \bar{A}^{\alpha\beta\gamma\delta} \left[\frac{1}{2}(\dot{u}_{\gamma\delta} + \ddot{u}_{\gamma\delta}) + \frac{\chi_3}{\bar{t}}(\dot{u}_{\gamma\delta} - \ddot{u}_{\gamma\delta} \right. \right. \\ &\quad \left. \left. + t W_{\gamma\delta}) + \frac{1}{2} W_{\gamma} W_{\delta} + \bar{Z}_{\gamma} W_{\delta} \right] + \bar{D}^{\alpha\beta} T \right\} d\chi_3 \quad (1-58a) \\ &= \bar{A}^{\alpha\beta\gamma\delta} \bar{t} \left[\frac{1}{2}(\dot{u}_{\gamma\delta} + \ddot{u}_{\gamma\delta}) + \frac{1}{2} W_{\gamma} W_{\delta} + \bar{Z}_{\gamma} W_{\delta} \right] + \bar{T}^{\alpha\beta}, \end{aligned}$$

where

$$\bar{T}^{\alpha\beta} = \int_{-\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2}} \bar{D}^{\alpha\beta} T d\chi_3, \quad (1-58b)$$

$$\bar{M}^{\alpha\beta} = \bar{A}^{\alpha\beta\gamma\delta} \frac{\bar{t}^2}{12} (\dot{u}_{\gamma\delta} - \ddot{u}_{\gamma\delta} + t W_{\gamma\delta}) + \bar{M}_T^{\alpha\beta}, \quad (1-58c)$$

$$\bar{M}_T^{\alpha\beta} = \int_{-\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2}} \bar{D}^{\alpha\beta} T X_3 dX_3, \quad (1-58d)$$

and

$$Q^{\alpha\beta} = \bar{A}^{\alpha\beta\gamma\delta} \frac{1}{2} [u'_\gamma - u''_\gamma + (t + \bar{t}) W_\gamma]. \quad (1-58e)$$

From Equation (1-53), it can be written that

$$\begin{aligned} N^{\alpha\beta} = & A^{\alpha\beta\gamma\delta} \left[t(u'_{\gamma\delta} + u''_{\gamma\delta} + 2 Z_{\gamma} W_{\delta} + W_{\gamma} W_{\delta}) \right] \\ & + \bar{A}^{\alpha\beta\gamma\delta} \bar{t} \left[\frac{1}{2}(u'_{\gamma\delta} + u''_{\gamma\delta}) + \frac{1}{2} W_{\gamma} W_{\delta} \right. \\ & \left. + Z_{\gamma} W_{\delta} \right] + T'^{\alpha\beta} + T''^{\alpha\beta} + \bar{T}^{\alpha\beta}, \end{aligned} \quad (1-59)$$

$$\begin{aligned} M^{\alpha\beta} = & -A^{\alpha\beta\gamma\delta} \frac{t^3}{6} W_{\gamma\delta} + \bar{A}^{\alpha\beta\gamma\delta} \frac{\bar{t}^2}{12} (u'_{\gamma\delta} - u''_{\gamma\delta} \\ & + t W_{\gamma\delta}) + M_T'^{\alpha\beta} + M_T''^{\alpha\beta} + \bar{M}_T^{\alpha\beta}. \end{aligned} \quad (1-60)$$

By substituting Equations (1-56), (1-57), (1-58), (1-59) and (1-60) into Equations (1-52) and (1-54), we obtain five simultaneous differential equations and boundary conditions in terms of the displacements u^I_α , u''_α and w .

B. Method B

Since the equilibrium equations obtained from method (A) have a rather complicated form in terms of the reference

displacements u^0_α , u''_α and w , it is desirable to convert these equations into a simpler form in order to solve the equilibrium equations. Different reference variables are chosen here to derive the equilibrium equations.

We take $\bar{u}_\alpha$, the displacement of the middle surface of the core, θ_α , the angular displacement of the cross section originally normal to the middle surface of the core, and w , the transverse deflection of the sandwich shell, as the reference variables.

a. Strain-Displacement Relations and Strain Energy of a Sandwich Structure

The same assumption made in method (A) is used for the core; i.e., planes normal to the middle surface of the core still remain plane after deformation. Consequently we have the displacements for the core expressed by

$$u_\alpha = \bar{u}_\alpha + \chi_3 \theta_\alpha, \quad (1-61)$$

$$\text{at } \chi_3 = \frac{\bar{t}}{2}; \quad u_\alpha = \bar{u}_\alpha + \frac{\bar{t}}{2} \theta_\alpha, \quad (1-62a)$$

$$\chi_3 = -\frac{\bar{t}}{2}; \quad u_\alpha = \bar{u}_\alpha - \frac{\bar{t}}{2} \theta_\alpha. \quad (1-62b)$$

Substituting Equation(1-61) into Equation (1-22b), we can express the strains in terms of the displacements

$$\bar{\epsilon}_{\alpha\beta} = \frac{1}{2}(\bar{u}_{\alpha/\beta} + \chi_3 \theta_{\alpha/\beta} + \bar{u}_{\beta/\alpha} + \chi_3 \theta_{\beta/\alpha} + \bar{z}_\alpha w_{/\beta} + \bar{z}_\beta w_{/\alpha} + w_{/\alpha} w_{/\beta}). \quad (1-63)$$

By Equation (1-39), we obtain the transverse shear strains

$$\bar{\epsilon}_{\alpha 3} = \frac{1}{2}(\theta_\alpha + w_{/\alpha}). \quad (1-64)$$

The variation of strain energy for the core can be obtained as

$$\begin{aligned} \delta \bar{U} &= \int_V \bar{\sigma}^{\alpha\beta} \delta \bar{\epsilon}_{\alpha\beta} dV + 2 \int_V \bar{\sigma}^{\alpha 3} \delta \bar{\epsilon}_{\alpha 3} dV \\ &= \int_A \left[\bar{N}^{\alpha\beta} \delta(\bar{u}_{\alpha/\beta} + \bar{z}_\alpha w_{/\beta} + \frac{1}{2} w_{/\alpha} w_{/\beta} \right. \\ &\quad \left. + \bar{M}^{\alpha\beta} \theta_{\alpha/\beta} + Q^{\alpha 3} \delta(\theta_\alpha + w_{/\alpha}) \right] dA, \end{aligned} \quad (1-65)$$

where $\bar{N}^{\alpha\beta}$, $\bar{M}^{\alpha\beta}$ and $Q^{\alpha 3}$ are defined by Equations (1-50a), (1-50b) and (1-50c), respectively.

For the upper facing, with $\frac{\bar{t}}{2} \leq x_3 \leq \frac{\bar{t}}{2} + t$, and applying the Kirchhoff-Love hypothesis, we have the displacements expressed as

$$u_\alpha = \bar{u}_\alpha + \frac{\bar{t}}{2} \theta_\alpha - (\chi_3 - \frac{\bar{t}}{2}) w_{/\alpha}. \quad (1-66)$$

From Equation (1-22b) the strains in the upper facing can be found in terms of the displacements

$$\begin{aligned} \varepsilon'_{\alpha\beta} = \frac{1}{2} \left[\bar{u}_{\alpha/\beta} + \frac{\bar{t}}{2} \theta_{\alpha/\beta} - \left(\chi_3 - \frac{\bar{t}}{2} \right) W_{\alpha/\beta} + \bar{u}_{\beta/\alpha} \right. \\ \left. + \frac{\bar{t}}{2} \theta_{\beta/\alpha} - \left(\chi_3 - \frac{\bar{t}}{2} \right) W_{\beta/\alpha} + Z_{\alpha} W_{\beta} + Z_{\beta} W_{\alpha} + W_{\alpha} W_{\beta} \right]. \end{aligned} \quad (1-67)$$

For the lower facing, with $-\frac{\bar{t}}{2} - t \leq x_3 \leq -\frac{\bar{t}}{2}$, and applying the Kirchhoff-Love hypothesis, we have the displacements expressed as

$$u_{\alpha} = \bar{u}_{\alpha} - \frac{\bar{t}}{2} \theta_{\alpha} - \left(\chi_3 + \frac{\bar{t}}{2} \right) W_{\alpha}. \quad (1-68)$$

For the strain-displacement relations of the lower facing, using Equations (1-22) and (1-68), we obtain

$$\begin{aligned} \varepsilon''_{\alpha\beta} = \frac{1}{2} \left[\bar{u}_{\alpha/\beta} - \frac{\bar{t}}{2} \theta_{\alpha/\beta} - 2 \left(\chi_3 + \frac{\bar{t}}{2} \right) W_{\alpha/\beta} + \bar{u}_{\beta/\alpha} \right. \\ \left. - \frac{\bar{t}}{2} \theta_{\beta/\alpha} + W_{\alpha} Z_{\beta} + Z_{\alpha} W_{\beta} + W_{\alpha} W_{\beta} \right]. \end{aligned} \quad (1-69)$$

The variation of strain energy for the upper facing is

$$\begin{aligned} \delta U' = \int_V \left\{ \sigma'^{\alpha\beta} \delta \left[\bar{u}_{\alpha/\beta} + \frac{\bar{t}}{2} \theta_{\alpha/\beta} - \left(\chi_3 - \frac{\bar{t}}{2} \right) W_{\alpha/\beta} \right. \right. \\ \left. \left. + Z_{\alpha} W_{\beta} + \frac{1}{2} W_{\alpha} W_{\beta} \right] \right\} dV \\ = \int_A \left\{ N'^{\alpha\beta} \delta \left(\bar{u}_{\alpha/\beta} + \frac{\bar{t}}{2} \theta_{\alpha/\beta} + Z_{\alpha} W_{\beta} + \frac{1}{2} W_{\alpha} W_{\beta} - M'^{\alpha\beta} W_{\alpha/\beta} \right) \right\} dA, \end{aligned} \quad (1-70)$$

where

$$N'^{\alpha\beta} = \int_{\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2}+t} \sigma'^{\alpha\beta} d\chi_3, \quad (1-71a)$$

$$M'^{\alpha\beta} = \int_{\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2}+t} \sigma'^{\alpha\beta} (\chi_3 - \frac{\bar{t}}{2}) d\chi_3. \quad (1-71b)$$

The variation of strain energy for the lower facing is

$$\delta U'' = \int_A \left\{ N''^{\alpha\beta} \delta \left(\bar{u}_{\alpha/\beta} - \frac{\bar{t}}{2} \theta_{\alpha/\beta} + \frac{\bar{t}}{2} w_{/\beta} + \frac{1}{2} w_{/\alpha} w_{/\beta} \right) - M''^{\alpha\beta} \delta w_{/\alpha\beta} \right\} dA, \quad (1-72)$$

where

$$N''^{\alpha\beta} = \int_{-\frac{\bar{t}}{2}-t}^{-\frac{\bar{t}}{2}} \sigma''^{\alpha\beta} d\chi_3, \quad (1-73a)$$

$$M''^{\alpha\beta} = \int_{-\frac{\bar{t}}{2}-t}^{-\frac{\bar{t}}{2}} \sigma''^{\alpha\beta} (\chi_3 + \frac{\bar{t}}{2}) d\chi_3. \quad (1-73b)$$

b. Derivation of the Equilibrium Equations

The total variation of the strain energy stored in the sandwich shell is

$$\begin{aligned}
\delta U_T &= \delta U' + \delta U'' + \delta \bar{U} \\
&= \int_A \left\{ N'^{\alpha\beta} \delta (\bar{U}_{\alpha/\beta} + \frac{\bar{E}}{2} \theta_{\alpha/\beta} + Z_{\alpha} W_{/\beta} + \frac{1}{2} W_{\alpha} W_{/\beta}) \right. \\
&\quad - M'^{\alpha\beta} \delta W_{\alpha/\beta} + N''^{\alpha\beta} \delta (\bar{U}_{\alpha/\beta} - \frac{\bar{E}}{2} \theta_{\alpha/\beta} + Z_{\alpha} W_{/\beta} + \frac{1}{2} W_{\alpha} W_{/\beta}) \\
&\quad - M''^{\alpha\beta} \delta W_{\alpha/\beta} + \bar{N}^{\alpha\beta} \delta (\bar{U}_{\alpha/\beta} + Z_{\alpha} W_{/\beta} + \frac{1}{2} W_{\alpha} W_{/\beta}) \\
&\quad \left. + \bar{M}^{\alpha\beta} \delta \theta_{\alpha/\beta} + Q^{\alpha\beta} \delta (\theta_{\alpha} + W_{/\alpha}) \right\} dA.
\end{aligned} \tag{1-74}$$

By Equation (1-43), the theorem of minimum potential energy yields

$$\begin{aligned}
&\int_A \left\{ N^{\alpha\beta} \delta \bar{U}_{\alpha/\beta} + \left[\frac{\bar{E}}{2} (N'^{\alpha\beta} - N''^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \delta \theta_{\alpha/\beta} \right. \\
&\quad + N^{\alpha\beta} (Z_{\alpha} \delta W_{/\beta} + W_{\alpha} \delta W_{/\beta}) - (M'^{\alpha\beta} + M''^{\alpha\beta}) \delta W_{\alpha/\beta} \\
&\quad \left. + Q^{\alpha\beta} \delta \theta_{\alpha} + Q^{\alpha\beta} \delta W_{/\alpha} - g \delta W \right\} dA = 0.
\end{aligned} \tag{1-75a}$$

By application of the divergence theorem, Equation (1-75a) becomes

$$\begin{aligned}
&-\int_A N^{\alpha\beta}_{/\beta} \delta \bar{U}_{\alpha} dA + \oint_S N^{\alpha\beta} \delta \bar{U}_{\alpha} \gamma_{\beta} dS - \int_A (\bar{M}^{\alpha\beta}_{/\beta} \\
&- Q^{\alpha\beta}) \delta \theta_{\alpha} dA + \oint_S \bar{M}^{\alpha\beta} \delta \theta_{\alpha} \gamma_{\beta} dS - \int_A \left\{ [N^{\alpha\beta} \right. \\
&\quad (Z_{/\alpha} + W_{\alpha})]_{/\beta} + (M'^{\alpha\beta} + M''^{\alpha\beta})_{\alpha/\beta} + Q^{\alpha\beta}_{/\alpha} + g \} \\
&\quad \delta W dA + \oint_S [N^{\alpha\beta} (W_{\alpha} + Z_{\alpha}) + (M'^{\alpha\beta} + M''^{\alpha\beta})_{/\alpha} \\
&\quad \left. - Q^{\beta\beta} \right] \delta W \gamma_{\beta} dS + \oint_S (M'^{\alpha\beta} + M''^{\alpha\beta}) \delta W_{\alpha} \gamma_{\beta} dS = 0,
\end{aligned} \tag{1-75b}$$

where

$$\hat{M}^{\alpha\beta} = \frac{\bar{E}}{2} (N'^{\alpha\beta} - N''^{\alpha\beta}) + \bar{M}^{\alpha\beta}. \tag{1-76}$$

Since $\delta \bar{u}_\alpha$, $\delta \theta_\alpha$ and δw are independent, from Equation (1-75b) we obtain a set of equilibrium equations

$$N^{\alpha\beta}_{/3} = 0, \quad (1-77a)$$

$$\hat{M}^{\alpha\beta}_{/3} - Q^{\alpha\beta} = 0, \quad (1-77b)$$

$$\begin{aligned} [N^{\alpha\beta}(Z_{/\alpha} + W_{/\alpha})]_{/3} + (M'^{\alpha\beta} + M''^{\alpha\beta})_{/3} \\ + Q^{\alpha\beta}_{/\alpha} + g = 0, \end{aligned} \quad (1-77c)$$

and boundary conditions in terms of stress resultants

$$N^{\alpha\beta} \gamma_\beta \delta \bar{u}_\alpha = 0, \quad (1-78a)$$

$$\hat{M}^{\alpha\beta} \gamma_\beta \delta \theta_\alpha = 0, \quad (1-78b)$$

$$[N^{\alpha\beta}(Z_{/\alpha} + W_{/\alpha}) + (M'^{\alpha\beta} + M''^{\alpha\beta})_{/\alpha} - Q^{\beta\alpha}] \gamma_\beta \delta w = 0, \quad (1-78c)$$

$$(M'^{\alpha\beta} + M''^{\alpha\beta}) \gamma_\beta \delta w_{/\alpha} = 0. \quad (1-78d)$$

The stress resultant-displacement relations can be obtained by using Equations (1-50a), (1-50b), (1-50c), (1-55a), (1-55b), (1-71a), (1-71b), (1-73a), and (1-73b)

$$N'^{\alpha\beta} = A^{\alpha\beta\gamma\delta} \left[t(\bar{u}_{\gamma\delta} + \frac{\bar{t}}{2} \theta_{\gamma\delta} + \frac{1}{2} w_{\gamma} w_{\delta} + \frac{1}{2} w_{\gamma} w_{\delta}) - \frac{t^2}{2} w_{\gamma\delta} \right] + T'^{\alpha\beta}, \quad (1-79)$$

where $T'^{\alpha\beta}$ is defined by Equation (1-56b),

$$N''^{\alpha\beta} = A^{\alpha\beta\gamma\delta} \left[t(\bar{u}_{\gamma\delta} - \frac{\bar{t}}{2} \theta_{\gamma\delta} + \frac{1}{2} w_{\gamma} w_{\delta} + \frac{1}{2} w_{\gamma} w_{\delta}) + \frac{t^2}{2} w_{\gamma\delta} \right] + T''^{\alpha\beta}, \quad (1-80)$$

where $T''^{\alpha\beta}$ is defined by Equation (1-57b),

$$M'^{\alpha\beta} = A^{\alpha\beta\gamma\delta} \left[\frac{t^2}{2} (\bar{u}_{\gamma\delta} + \frac{\bar{t}}{2} \theta_{\gamma\delta} + \frac{1}{2} w_{\gamma} w_{\delta} + \frac{1}{2} w_{\gamma} w_{\delta}) - \frac{t^3}{3} w_{\gamma\delta} \right] + M_T'^{\alpha\beta}, \quad (1-81a)$$

with

$$M_T'^{\alpha\beta} = \int_{\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2} + t} D^{\alpha\beta} T'(\chi_3 - \frac{\bar{t}}{2}) d\chi_3, \quad (1-81b)$$

$$M_T''^{\alpha\beta} = \int_{\frac{\bar{t}}{2}}^{\frac{\bar{t}}{2} + t} D^{\alpha\beta} T''(\chi_3 + \frac{\bar{t}}{2}) d\chi_3, \quad (1-82a)$$

$$M''^{\alpha\beta} = -A^{\alpha\beta\gamma\delta} \left[\frac{t^2}{2} (\bar{u}_{\gamma\delta} - \frac{\bar{t}}{2} \theta_{\gamma\delta} + \frac{1}{2} w_{\gamma} w_{\delta} + \frac{1}{2} w_{\gamma} w_{\delta}) + \frac{t^3}{2} w_{\gamma\delta} \right] + M_T''^{\alpha\beta}, \quad (1-82b)$$

$$\bar{N}^{\alpha\beta} = \bar{A}^{\alpha\beta\gamma\delta} \bar{t} (\bar{u}_{\gamma\delta} + \bar{z}_{\gamma} w_{\delta} + \frac{1}{2} w_{\gamma} w_{\delta}) + \bar{T}^{\alpha\beta}, \quad (1-83)$$

where $\bar{T}^{\alpha\beta}$ is defined by Equation (1-58b),

$$\bar{M}^{\alpha\beta} = \bar{A}^{\alpha\beta\gamma\delta} \frac{\bar{t}^2}{12} \theta_{\gamma\delta} + \bar{M}_T^{\alpha\beta}, \quad (1-84)$$

where $\bar{M}_T^{\alpha\beta}$ is defined by Equation (1-58d), and

$$Q^{\alpha\beta} = A^{\alpha\beta\gamma\delta} \frac{\bar{t}}{2} (\theta_{\gamma} + w_{\gamma}). \quad (1-85)$$

Also, from Equation (1-53a),

$$\begin{aligned} N^{\alpha\beta} = & A^{\alpha\beta\gamma\delta} t (2 \bar{u}_{\gamma\delta} + 2 \bar{z}_{\gamma} w_{\delta} + w_{\gamma} w_{\delta}) + \\ & \bar{A}^{\alpha\beta\gamma\delta} \bar{t} (\bar{u}_{\gamma\delta} + \bar{z}_{\gamma} w_{\delta} + \frac{1}{2} w_{\gamma} w_{\delta}) + \\ & T'^{\alpha\beta} + T''^{\alpha\beta} + \bar{T}^{\alpha\beta}, \end{aligned} \quad (1-86)$$

and, from Equation (1-76),

$$\begin{aligned} \hat{M}^{\alpha\beta} = & \frac{\bar{t}}{2} \left\{ A^{\alpha\beta\gamma\delta} (t \bar{t} \theta_{\gamma\delta} - t^2 w_{\gamma\delta}) + T'^{\alpha\beta} \right. \\ & \left. - T''^{\alpha\beta} \right\} + \bar{A}^{\alpha\beta\gamma\delta} \frac{\bar{t}^2}{12} \theta_{\gamma\delta} + \bar{M}_T^{\alpha\beta}. \end{aligned} \quad (1-87)$$

By substituting Equations (1-86), (1-87), (1-85), (1-81a) and (1-82a) into Equations (1-77a), (1-77b) and (1-77c), we have the corresponding equilibrium equations in terms of the displacement variables $\bar{u}_{\alpha}$, θ_{α} and w . The boundary conditions can be obtained similarly. The practical application of these equations will be discussed in Chapter II.

CHAPTER II

PRACTICAL APPLICATION

The above derivation is general enough to apply to any coordinate system. From a practical viewpoint the coordinate system is chosen in accordance with the actual problem in order to save time and labor. Here, we consider an axisymmetrical case of the sandwich shallow shell. In this case it is most desirable to choose cylindrical coordinates r , θ , and z . Also, due to the axisymmetrical nature, the displacement is independent of the coordinate θ . In the following, we are going to convert the tensorial components into physical components and the equations in tensorial form into engineering form. We list the non-zero components of the metric tensor and Christoffel symbols in a cylindrical coordinate system. Thus

$$\begin{aligned} e_{11} &= 1, & e_{22} &= r^2, \\ e'' &= 1, & e^{22} &= \frac{1}{r^2}, \\ \Gamma_{22}^1 &= -r, & \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{r}. \end{aligned} \quad (2-1)$$

1. Physical Components

A. Physical Components of the Elastic Coefficients and Thermal Coefficients

Like most problems in shell theory the elastic coefficients are obtained from a generalized plane-stress assumption.

a. For Isotropic Material

$$A'''' \approx \frac{E}{1-\nu^2},$$

$$A''^{1122} \approx \frac{E\nu}{1-\nu^2} = A''^{2211}, \quad (2-2)$$

$$A''^{1212} = A''^{2121} = A''^{2112} = A''^{1221} \approx \frac{E}{2(1+\nu)},$$

$$A''^{2222} \approx \frac{E}{1-\nu^2},$$

$$D'' \approx -\frac{E\alpha}{1-\nu} \quad (2-3)$$

$$D''^{22} \approx -\frac{E\alpha}{1-\nu}$$

where E is Young's modulus and ν is Poisson's ratio.

b. For Orthotropic Material (30)

$$\bar{A}'''' \approx \frac{E_r}{1-\nu_{or}\nu_{ro}}, \quad \bar{A}''^{2222} = \frac{E_a}{1-\nu_{or}\nu_{ro}},$$

$$\bar{A}''^{1122} = \bar{A}''^{2211} \approx \frac{E_r\nu_{or}}{1-\nu_{or}\nu_{ro}} = \frac{E_a\nu_{ro}}{1-\nu_{or}\nu_{ro}},$$

$$\bar{A}''^{1212} = 2G_3, \quad \bar{A}''^{1313} \approx 2G_2, \quad (2-4)$$

$$\bar{A}''^{2323} = 2G_1,$$

$$\begin{aligned}\bar{D}'' &= - \frac{E_r(\alpha_r + \gamma_{ar}\alpha_a)}{1 - \gamma_{10}\gamma_{0r}}, \\ \bar{D}^{22} &= - \frac{E_a(\alpha_a + \gamma_{ar}\alpha_r)}{1 - \gamma_{ra}\gamma_{ar}}.\end{aligned}\quad (2-5)$$

B. Physical Componets of the Derivatives

For the first covariant derivatives it has been shown in (27) that

$$N^{\alpha\beta}_{\gamma} = N^{\alpha\beta}_{,\gamma} + \Gamma^{\alpha}_{\delta\gamma} N^{\delta\beta} + \Gamma^{\beta}_{\delta\gamma} N^{\alpha\delta}. \quad (2-6)$$

From Equation (2-6), we have

$$\begin{aligned}N''_{11} &= N''_{,1}, \\ N'^{12}_{11} &= N'^{12}_{,11} + \Gamma^{12}_{21} N'^2, \\ N'^{12}_{12} &= \Gamma'^1_{22} N^{22} + \Gamma^2_{12} N'', \\ N''_{12} &= 2 \Gamma'^1_{22} N^{21}, \\ N^{22}_{11} &= N^{22}_{,11} + 2 \Gamma^2_{21} N^{22}, \\ N^{22}_{12} &= 2 \Gamma^2_{12} N'^2.\end{aligned}\quad (2-7)$$

For the second covariant derivatives we obtain

$$M^{\alpha\beta}_{\gamma\delta} = (M^{\alpha\beta}_{\gamma})_{,\delta} + \Gamma^{\alpha}_{\omega\delta} M^{\omega\beta}_{\gamma} + \Gamma^{\beta}_{\omega\delta} M^{\alpha\omega}_{\gamma} - \Gamma^{\omega}_{\gamma\delta} M^{\alpha\beta}_{\omega}. \quad (2-8)$$

Applying formulae (2-6), (2-7) and (2-8), we have

$$\begin{aligned} M''_{/1} &= M''_{,11} , \\ M'^2_{/12} &= \Gamma'_{22} M^{22}_{/1} + \Gamma^2_{12} M''_{/1} - \Gamma^2_{12} M'^2_{/2} , \\ M^{22}_{/22} &= 2 \Gamma^2_{12} M'^2_{/2} - \Gamma'^2_{22} M^{22}_{/1} , \end{aligned} \quad (2-9)$$

$$\begin{aligned} W_{/\alpha} &= W_{, \alpha} \\ W_{/\alpha\beta} &= W_{, \alpha\beta} - \Gamma^{\gamma}_{\alpha\beta} W_{/\gamma} \end{aligned} \quad (2-10)$$

From Equation (2-10), we have

$$\begin{aligned} W_{/1} &= W_{,1} , \\ W_{/22} &= - \Gamma'^1_{22} W_{/1} , \\ W_{/11} &= W_{,11} . \end{aligned} \quad (2-11)$$

Similarly Equation (2-11) can also be applied to Z.

Also, we have

$$V_{\alpha/\beta} = V_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} V_{/\gamma} . \quad (2-12)$$

Hence,

$$\begin{aligned} u_{/1} &= u_{,1} , \\ u_{/22} &= - \Gamma'^1_{22} u_{/1} . \end{aligned} \quad (2-13)$$

C. Physical Components of the Stress Resultants

The relation between physical components and tensorial components is expressed as follows

$$P_{(\alpha\beta)} \approx \sqrt{e_{\alpha\alpha}} \sqrt{e_{\beta\beta}} P^{\alpha\beta}, \quad (2-14)$$

$$P_{(\alpha\beta)} \approx \sqrt{e^{\alpha\alpha}} \sqrt{e^{\beta\beta}} P_{\alpha\beta}.$$

From Equation (2-14), we convert all the tensorial components into the physical components, thus

$$\begin{aligned} N'' &\approx N_r, & N'^{22} &\approx \frac{1}{r^2} N_\theta, \\ Q'^3 &\approx Q_r, & M'' &\approx M_r, \\ M'^{22} &\approx \frac{1}{r^2} M_\theta. \end{aligned} \quad (2-15)$$

2. Equilibrium Equations and Boundary Conditions in Engineering Form

We use Equations (2-7), (2-9) and (2-15) to obtain

$$\begin{aligned} N'^{\beta}_{\beta} &= \frac{dN_r}{dr} - \frac{1}{r} N_\theta + \frac{1}{r} N_r, \\ M'^{\alpha\beta}_{\alpha\beta} &= \frac{d^2 M_r}{dr^2} + \frac{2}{r} \frac{dM_r}{dr} - \frac{1}{r} \frac{dM_\theta}{dr}. \end{aligned} \quad (2-16)$$

A. For Method A

Using Equation (2-16), we can have, by introducing two new variables $\xi_\alpha = \frac{1}{2} (u'_\alpha + u''_\alpha)$ and $\psi_\alpha = \frac{1}{2}(u'_\alpha - u''_\alpha)$, the equilibrium Equation (1-52) written in terms of the engineering form of the stress resultants. Thus

$$\begin{aligned} r \frac{dN_r}{dr} + N_r - N_\alpha &= 0, \\ r \frac{d\hat{M}_r}{dr} + \hat{M}_r - \hat{M}_\alpha - rQ_r &= 0, \\ \frac{d}{dr} \left[r N_r \frac{d(W+Z)}{dr} \right] + \hat{t} \frac{d}{dr} (r Q_r) + \bar{g} r \\ + \left[r \frac{d^2 \bar{M}_r}{dr^2} - \frac{d\bar{M}_\alpha}{dr} + 2 \frac{d\bar{M}_r}{dr} - \hat{t} \left(r \frac{d^2 \bar{M}_r}{dr^2} \right. \right. \\ \left. \left. - \frac{d\bar{M}_\alpha}{dr} + 2 \frac{d\bar{M}_r}{dr} \right) \right] &= 0, \end{aligned} \quad (2-17a)$$

where

$$\begin{aligned} \hat{M}_r &= \bar{M}_r + \frac{\bar{t}}{2} (N'_r - N''_r), \\ \hat{M}_\alpha &= \bar{M}_\alpha + \frac{\bar{t}}{2} (N'_\alpha - N''_\alpha). \end{aligned} \quad (2-17b)$$

a. Equilibrium Equations in Terms of Physical Displacements

The stress resultants can be expressed in terms of the engineering displacements as follows

$$(2-189) \quad \cdot \left(\frac{1}{M} W \right) + \left[\left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \frac{1}{T} \right) \frac{1}{T} + \left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \right. \right. \\ \left. \left. \frac{1}{M} \frac{P}{T} \right) \frac{1}{T} \right] \frac{1}{T} + \left[\frac{1}{T} + \frac{1}{M} \frac{P}{T} \right] \frac{1}{T} = \frac{1}{M} W$$

$$(2-187) \quad \cdot \left(\frac{1}{M} W \right) + \left[\left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \frac{1}{T} \right) \frac{1}{T} + \right. \\ \left. \frac{1}{M} \frac{P}{T} \right] \frac{1}{T} + \left[\frac{1}{T} + \frac{1}{M} \frac{P}{T} \right] \frac{1}{T} = \frac{1}{M} W$$

$$(2-186) \quad \cdot \left[\frac{1}{M} \frac{P}{T} \left(\frac{Z}{T} + \frac{1}{T} \right) \frac{1}{T} + \frac{1}{T} \right] \frac{1}{T} = \frac{1}{M} W$$

$$(2-184) \quad \cdot \left(\frac{1}{M} W \right) + \left[\left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \frac{1}{T} \right) \frac{1}{T} + \left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \right. \right. \\ \left. \left. \frac{1}{M} \frac{P}{T} \right) \frac{1}{T} \right] \frac{1}{T} + \left[\frac{1}{T} + \frac{1}{M} \frac{P}{T} \right] \frac{1}{T} = \frac{1}{M} W$$

$$(2-183) \quad \cdot \left(\frac{1}{M} W \right) + \left[\left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \frac{1}{T} \right) \frac{1}{T} + \left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \right. \right. \\ \left. \left. \frac{1}{M} \frac{P}{T} \right) \frac{1}{T} \right] \frac{1}{T} + \left[\frac{1}{T} + \frac{1}{M} \frac{P}{T} \right] \frac{1}{T} = \frac{1}{M} W$$

$$(2-182) \quad \cdot \left(\frac{1}{M} W \right) + \left[\left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \frac{1}{T} \right) \frac{1}{T} + \left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \right. \right. \\ \left. \left. \frac{1}{M} \frac{P}{T} \right) \frac{1}{T} \right] \frac{1}{T} + \left[\frac{1}{T} + \frac{1}{M} \frac{P}{T} \right] \frac{1}{T} = \frac{1}{M} W$$

$$(2-181) \quad \cdot \left(\frac{1}{M} W \right) + \left[\left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \frac{1}{T} \right) \frac{1}{T} + \left(\frac{1}{M} \frac{P}{T} \frac{Z}{T} + \right. \right. \\ \left. \left. \frac{1}{M} \frac{P}{T} \right) \frac{1}{T} \right] \frac{1}{T} + \left[\frac{1}{T} + \frac{1}{M} \frac{P}{T} \right] \frac{1}{T} = \frac{1}{M} W$$

where

$$T_r = T_r' + T_r'' + \bar{T}_r, \quad (2-18h)$$

$$T_\alpha = T_\alpha' + T_\alpha'' + \bar{T}_\alpha, \quad (2-18i)$$

$$(M_T)_r = (M_T')_r + (M_T'')_r + (\bar{M}_T)_r, \quad (2-18j)$$

$$(M_T)_\alpha = (M_T')_\alpha + (M_T'')_\alpha + (\bar{M}_T)_\alpha, \quad (2-18k)$$

$$(\hat{M}_T)_r = (\bar{M}_T)_r + \frac{\bar{E}}{2} (T_r' - T_r''), \quad (2-18l)$$

$$(\hat{M}_T)_\alpha = (\bar{M}_T)_\alpha + \frac{\bar{E}}{2} (T_\alpha' - T_\alpha''). \quad (2-18m)$$

By substituting Equations (2-18) into Equations (2-17) we have the equilibrium equations in terms of the engineering displacements

$$\begin{aligned} & r \left(\frac{d^2 \xi_r}{dr^2} + \frac{d^2 \xi}{dr^2} \frac{dW}{dr} + \frac{d\xi}{dr} \frac{d^2 W}{dr^2} + \frac{dW}{dr} \frac{d^2 W}{dr^2} \right) + \frac{d\xi_r}{dr} \\ & + A_1' \left[\frac{1}{2} \left(\frac{dW}{dr} \right)^2 + \frac{d\xi}{dr} \frac{dW}{dr} \right] - A_2' \frac{\xi_r}{r} + A_{10}' \left(r \frac{dT_r}{dr} \right. \\ & \left. + T_r - T_\alpha \right) = 0, \end{aligned} \quad (2-19a)$$

$$\begin{aligned} & r \frac{d^2 \psi_r}{dr^2} + \frac{d\psi_r}{dr} - \left(\frac{A_4'}{r} + A_5' r \right) \psi_r + A_3' \left(r \frac{d^3 W}{dr^3} + \frac{d^2 W}{dr^2} \right) \\ & - \left(\frac{A_6'}{r} + A_7' r \right) \frac{dW}{dr} + \frac{1}{A_{15}'} \left[r \frac{d(\hat{M}_T)_r}{dr} + (\hat{M}_T)_r \right. \\ & \left. - (\hat{M}_T)_\alpha \right] = 0, \end{aligned} \quad (2-19b)$$

$$\begin{aligned}
& \left[\frac{d\tilde{\zeta}_r}{dr} + \frac{d\tilde{z}}{dr} \frac{dW}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 + A'_8 \frac{\tilde{\zeta}_r}{r} + A'_{10} T_0 \right] \times \\
& \left(r \frac{d^2 W}{dr^2} + r \frac{d^2 \tilde{z}}{dr^2} \right) + \left[A'_9 \left(\frac{d\tilde{\zeta}_r}{dr} + \frac{dW}{dr} \frac{d\tilde{z}}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right) \right. \\
& \left. + A'_2 \frac{\tilde{\zeta}_r}{r} + A'_{10} T_0 \right] \left(\frac{dW}{dr} + \frac{d\tilde{z}}{dr} \right) + A'_{10} g r + \\
& A'_{11} \left(r \frac{d^2 W}{dr^2} + 2 \frac{d^3 W}{dr^3} \right) + A'_{13} \left(\frac{1}{r^2} \frac{dW}{dr} - \frac{1}{r} \frac{d^2 W}{dr^2} \right) \\
& + A'_{12} \left(r \frac{d^2 \psi_r}{dr^2} + 2 \frac{d^2 \psi_r}{dr^2} \right) - A'_{14} \left(\frac{1}{r} \frac{d\psi_r}{dr} - \frac{\psi_r}{r^2} \right) \\
& + A'_{10} F_T(r) = 0,
\end{aligned}
\tag{2-19c}$$

where

$$A'_1 = \left[\frac{2Et}{1+\nu} + \frac{E_r \bar{t} (1-\nu_{ar})}{1-\nu_{ra} \nu_{ar}} \right] A'_{10},
\tag{2-19d}$$

$$A'_2 = \left[\frac{2Et}{1-\nu^2} + \frac{E_a \bar{t}}{1-\nu_{ra} \nu_{ar}} \right] A'_{10},
\tag{2-19e}$$

$$A'_3 = \frac{\frac{E_r t \bar{t}^2}{12(1-\gamma_{ra}\gamma_{ar})}}{\frac{Et\bar{t}}{1-\gamma^2} + \frac{E_r \bar{t}^2}{6(1-\gamma_{ra}\gamma_{ar})}} \quad , \quad (2-19f)$$

$$A'_4 = \frac{\frac{E_a \bar{t}^2}{6(1-\gamma_{ra}\gamma_{ar})} + \frac{Et\bar{t}}{1-\gamma^2}}{\frac{Et\bar{t}}{1-\gamma^2} + \frac{E_r \bar{t}^2}{6(1-\gamma_{ra}\gamma_{ar})}} \quad , \quad (2-19g)$$

$$A'_5 = \frac{2G_1}{\frac{Et\bar{t}}{1-\gamma^2} + \frac{E_r \bar{t}^2}{6(1-\gamma_{ra}\gamma_{ar})}} \quad , \quad (2-19h)$$

$$A'_6 = \frac{\frac{E_a t \bar{t}^2}{12(1-\gamma_{ra}\gamma_{ar})}}{\frac{Et\bar{t}}{1-\gamma^2} + \frac{E_r \bar{t}^2}{6(1-\gamma_{ra}\gamma_{ar})}} \quad , \quad (2-19i)$$

$$A'_7 = \frac{A'_5 (t + \bar{t})}{2} \quad (2-19j)$$

$$A'_8 = \left(\frac{2E\gamma t}{1-\gamma^2} + \frac{E_r \gamma_{ar} \bar{t}}{1-\gamma_{ra}\gamma_{ar}} \right) A'_{10} \quad , \quad (2-19k)$$

$$A'_9 = \left(\frac{2Et}{1-\nu^2} + \frac{E_r \bar{t}}{1-\nu_{ra}\nu_{\theta r}} \right) A'_{10} \quad (2-191)$$

$$A'_{10} = \frac{1}{\frac{2Et}{1-\nu^2} + \frac{E_r \bar{t}}{1-\nu_{ra}\nu_{\theta r}}} \quad (2-19m)$$

$$A'_{11} = \left[\frac{Et\bar{t}^2}{12(1-\nu_{ra}\nu_{\theta r})} - \frac{Et^3}{6(1-\nu^2)} \right] A'_{10} \quad (2-19n)$$

$$A'_{12} = \left[\frac{Et(t+\bar{t})}{1-\nu^2} + \frac{E_r \bar{t}^2}{6(1-\nu_{ra}\nu_{\theta r})} \right] A'_{10} \quad (2-19o)$$

$$A'_{13} = \left[\frac{E_r t \bar{t}^2}{12(1-\nu_{ra}\nu_{\theta r})} - \frac{Et^3}{6(1-\nu^2)} \right] A'_{10} \quad (2-19p)$$

$$A'_{14} = \left[\frac{E_r \bar{t}^2}{6(1-\nu_{ra}\nu_{\theta r})} + \frac{Et(t+\bar{t})}{1-\nu^2} \right] A'_{10} \quad (2-19q)$$

$$\begin{aligned} F_T(r) = & \hat{t} \frac{d}{dr} \left[r \frac{d(\hat{M}_T)_r}{dr} + (\hat{M}_T)_r - (\hat{M}_T)_\theta \right] \\ & + r \frac{d^2(M_T)_r}{dr^2} - \frac{d(M_T)_\theta}{dr} + 2 \frac{d(M_T)_r}{dr} \\ & - \hat{t} \left[r \frac{d^2(\bar{M}_T)_r}{dr^2} - \frac{d(\bar{M}_T)_\theta}{dr} + 2 \frac{d(\bar{M}_T)_r}{dr} \right]. \end{aligned} \quad (2-19r)$$

b. Boundary Conditions in Terms of Physical Displacements

The boundary conditions can be expressed by

$$N_r \delta \xi_r = 0, \quad (2-20a)$$

$$\bar{M}_r \delta \psi_r = 0, \quad (2-20b)$$

$$\left[r N_r \frac{d}{dr}(W+Z) + r \frac{dM_r}{dr} + M_r - M_\alpha - \hat{e} \left(r \frac{d\bar{M}_r}{dr} + \bar{M}_r - \bar{M}_\alpha \right) + \hat{e} r Q_r \right] \delta W = 0. \quad (2-20c)$$

$$(M_r - \hat{e} \bar{M}_r) \delta \left(\frac{dW}{dr} \right) = 0. \quad (2-20d)$$

By using Equations (2-18a), (2-18b), (2-18c), (2-18d), (2-18e), (2-18f) and (2-18g), we obtain the boundary conditions in terms of engineering displacements

$$\left\{ \frac{A_9'}{A_{10}'} \left[\frac{d\xi_r}{dr} + \frac{dZ}{dr} \frac{dW}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right] + \frac{A_8'}{A_{10}'} \frac{\xi_r}{r} + T_r \right\} \delta \xi_r = 0, \quad (2-21a)$$

$$\left\{ A_{15}' \frac{d\psi_r}{dr} + A_{16}' \frac{\psi_r}{r} + A_3' A_{15}' \left(\frac{d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right) + (\bar{M}_T)_r \right\} \delta \psi_r = 0, \quad (2-21b)$$

$$\begin{aligned}
& \left\{ \left\{ \frac{A'_2}{A'_{10}} \left[\frac{d\xi_r}{dr} + \frac{dz}{dr} \frac{dW}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right] + \frac{A'_2}{A'_{10}} \frac{\xi_r}{r} \right. \right. \\
& \quad \left. \left. + T_r \right\} r \frac{d}{dr} (W+Z) - A'_{21} \left(r \frac{d^3 W}{dr^3} + \frac{d^2 W}{dr^2} \right) + \right. \\
& \quad \frac{A'_{13}}{r} \frac{dW}{dr} - A'_{17} \left(r \frac{d^2 \psi_r}{dr^2} + \frac{d\psi_r}{dr} \right) + \left(\frac{A'_{18}}{r} + A'_{19} r \right) \psi_r \\
& \quad + A'_{20} r \frac{dW}{dr} + \left[r \frac{d(M_T)_r}{dr} + (M_T)_r - (M_T)_\theta \right] \\
& \quad \left. - \hat{t} \left[r \frac{d(\bar{M}_T)_r}{dr} + (\bar{M}_T)_r - (\bar{M}_T)_\theta \right] \right\} \delta W = 0 \quad (2-21c)
\end{aligned}$$

$$\begin{aligned}
& \left[A'_{21} \frac{d^2 W}{dr^2} + A'_{22} \frac{1}{r} \frac{dW}{dr} + A'_{17} \left(\frac{d\psi_r}{dr} + \frac{\psi_r}{r} \right) \right. \\
& \quad \left. - (M_T)_r + \hat{t} (\bar{M}_T)_r \right] \delta \left(\frac{dW}{dr} \right) = 0, \quad (2-21d)
\end{aligned}$$

where

$$A'_{15} = \frac{Et\bar{t}}{1-\nu^2} + \frac{E_r \bar{t}^2}{6(1-\nu_{\theta r}\nu_{\theta r})}, \quad (2-22a)$$

$$A'_{16} = \frac{Et\bar{t}\nu}{1-\nu^2} + \frac{E_r \nu_{\theta r} \bar{t}^2}{6(1-\nu_{\theta r}\nu_{\theta r})}, \quad (2-22b)$$

$$A'_{17} = \frac{E_r t \bar{t}}{6(1-\nu_{rs}\nu_{sr})} , \quad (2-22c)$$

$$A'_{18} = \frac{E_r t \bar{t}}{6(1-\nu_{rs}\nu_{sr})} , \quad (2-22d)$$

$$A'_{19} = 2G_r \left(\frac{t + \bar{t}}{\bar{t}} \right) , \quad (2-22e)$$

$$A'_{20} = \frac{G_r (t + \bar{t})^2}{\bar{t}} , \quad (2-22f)$$

$$A'_{21} = \frac{E t^3}{6(1-\nu^2)} + \frac{E_r \bar{t} t^2}{12(1-\nu_{rs}\nu_{sr})} , \quad (2-22g)$$

$$A'_{22} = \frac{E \nu t^3}{6(1-\nu^2)} + \frac{E_r \nu_{sr} \bar{t} t^2}{12(1-\nu_{rs}\nu_{sr})} . \quad (2-22h)$$

B. For Method B

Substituting Equation (2-16) into the equilibrium Equations (1-77), we have in terms of the stress resultants in engineering form

$$r \frac{dN_r}{dr} + N_r - N_\theta = 0 , \quad (2-23a)$$

$$r \frac{dM_r}{dr} + \hat{M}_r - \hat{M}_\theta - r Q_r = 0 , \quad (2-23b)$$

$$\begin{aligned}
N_r \left(\frac{d^2 W}{dr^2} + \frac{d^2 Z}{dr^2} \right) + \frac{N_\theta}{r} \left(\frac{dW}{dr} + \frac{dZ}{dr} \right) + \frac{d^2}{dr^2} (M_r' \\
+ M_r'') + \frac{2}{r} \frac{d}{dr} (M_r' + M_r'') - \frac{1}{r} \frac{d}{dr} (M_\theta' + M_\theta'') + \frac{1}{r} \frac{d}{dr} (r Q_r) \\
+ f = 0.
\end{aligned} \quad (2-23c)$$

The boundary conditions become

$$N_r \delta \bar{u}_r = 0, \quad (2-24a)$$

$$\hat{M}_r \delta \theta_r = 0, \quad (2-24b)$$

$$\left[N_r \left(\frac{dW}{dr} + \frac{dZ}{dr} \right) + \frac{d}{dr} (M_r' + M_r'') + \frac{1}{r} (M_r' + M_r'' - M_\theta' - M_\theta'') - Q_r \right] \delta W = 0, \quad (2-24c)$$

$$(M_r' + M_r'') \delta \left(\frac{dW}{dr} \right) = 0. \quad (2-24d)$$

a. Equilibrium Equations in Terms of the Physical Displacements

By a similar procedure, we are able to express the stress resultants and moments in terms of engineering displacements as

$$\begin{aligned}
N_r = \frac{2Et}{1-\nu^2} \left[\frac{d\bar{u}_r}{dr} + \frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 + \nu \frac{\bar{u}_r}{r} \right] \\
+ \frac{E_r \bar{e}}{1-\nu_{ar} \nu_{or}} \left[\frac{d\bar{u}_r}{dr} + \frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right. \\
\left. + \nu_{or} \frac{\bar{u}_r}{r} \right] + T_r, \quad (2-25a)
\end{aligned}$$

where

$$T_r = T_r' + T_r'' + \bar{T}_r,$$

$$T_\theta = T_\theta' + T_\theta'' + \bar{T}_\theta.$$

$$N_\theta = \frac{2Et}{1-\nu^2} \left\{ \nu \left[\frac{d\bar{u}_r}{dr} + \frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right] + \frac{\bar{u}_r}{r} \right\} + \frac{E_\theta \bar{t}}{1-\nu_{\theta r} \nu_{\theta r}} \left\{ \nu_{\theta r} \left[\frac{d\bar{u}_r}{dr} + \frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right] + \frac{\bar{u}_r}{r} \right\} + T_\theta. \quad (2-25b)$$

From Equation (1-87),

$$\hat{M}_r = \frac{Et}{2(1-\nu^2)} \left[t\bar{t} \frac{d\theta_r}{dr} - t^2 \frac{d^2W}{dr^2} + \nu \left(t\bar{t} \frac{\theta_r}{r} - t^2 \frac{1}{r} \frac{dW}{dr} \right) \right] + \frac{E_r \bar{t}^3}{12(1-\nu_{\theta r} \nu_{\theta r})} \left(\frac{d\theta_r}{dr} + \nu_{\theta r} \frac{\theta_r}{r} \right) + (M_T)_r, \quad (2-25c)$$

$$\hat{M}_\theta = \frac{Et}{2(1-\nu^2)} \left[\nu \left(t\bar{t} \frac{d\theta_r}{dr} - t^2 \frac{d^2W}{dr^2} \right) + \left(t\bar{t} \frac{\theta_r}{r} - \frac{t^2}{r} \frac{dW}{dr} \right) \right] + \frac{E_\theta \bar{t}^3}{12(1-\nu_{\theta r} \nu_{\theta r})} \left(\nu_{\theta r} \frac{d\theta_r}{dr} + \frac{\theta_r}{r} \right) + (M_T)_\theta. \quad (2-25d)$$

From Equation (1-85)

$$Q_r = G_r \bar{t} \left(\theta_r + \frac{dW}{dr} \right), \quad (2-25e)$$

and from Equations (1-81a) and (1-82a),

$$M_r' + M_r'' = \frac{Et^2}{2(1-\nu^2)} \left[\bar{t} \frac{d\theta_r}{dr} - \frac{4}{3} t \frac{d^2W}{dr^2} + \nu \left(\bar{t} \frac{\theta_r}{r} - \frac{4}{3} \frac{t}{r} \frac{dW}{dr} \right) \right] + (M_T)' + (M_T)'', \quad (2-26a)$$

$$\begin{aligned}
M'_Q + M''_Q &= \frac{E t^2}{2(1-\nu^2)} \left[\bar{t} \frac{d\theta_r}{dr} - \frac{4}{3} t \frac{d^2 W}{dr^2} + \right. \\
&\quad \left. \nu \left(\bar{t} \frac{\theta_r}{r} - \frac{4}{3} t \frac{dW}{dr} \right) \right] + (M'_T)_r \\
&\quad + (M''_T)_r .
\end{aligned}
\tag{2-26b}$$

Substituting Equations (2-25) and (2-26) into Equations (2-23) we obtain the equilibrium equations in terms of the engineering displacements as

$$\begin{aligned}
&A_1 \left[r \frac{d^2 \bar{u}_r}{dr^2} + \frac{d\bar{u}_r}{dr} + r \left(\frac{d^2 W}{dr^2} \frac{dZ}{dr} + \frac{dW}{dr} \frac{d^2 Z}{dr^2} + \right. \right. \\
&\quad \left. \left. \frac{dW}{dr} \frac{d^2 W}{dr^2} \right) \right] - A_2 \frac{\bar{u}_r}{r} + A_3 \left[\frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right] \\
&+ r \frac{dT_r}{dr} + T_r - T_Q = 0 ,
\end{aligned}
\tag{2-27a}$$

$$\begin{aligned}
&A_4 \left(r \frac{d^2 \theta_r}{dr^2} + \frac{d\theta_r}{dr} \right) - A_5 \frac{\theta_r}{r} - A_6 \left(r \frac{d^3 W}{dr^3} \right. \\
&+ \frac{d^2 W}{dr^2} - \frac{1}{r} \frac{dW}{dr} \left. \right) - A_7 \left(r \theta_r + r \frac{dW}{dr} \right) + r \frac{d(\hat{M}_{T_L})}{dr} \\
&+ (\hat{M}_T)_r - (\hat{M}_T)_Q = 0 ,
\end{aligned}
\tag{2-27b}$$

$$\left\{ A_1 \left[\frac{d\bar{u}_r}{dr} + \frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 + \frac{\bar{u}_r}{r} \right] - A_3 \frac{\bar{u}_r}{r} + \right.$$

$$\begin{aligned}
& + T_r \left\{ \left(r \frac{d^2 W}{dr^2} + r \frac{d^2 Z}{dr^2} \right) + \left\{ (A_1 - A_3) \left[\frac{d\bar{U}_r}{dr} + \right. \right. \right. \\
& + \left. \frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right] + A_2 \frac{\bar{U}_r}{r} + T_\alpha \left\{ \left(\frac{dW}{dr} + \frac{dZ}{dr} \right) \right. \\
& - A_8 \left(r \frac{d^4 W}{dr^4} + 2 \frac{d^3 W}{dr^3} - \frac{1}{r} \frac{d^2 W}{dr^2} + \frac{1}{r^2} \frac{dW}{dr} \right) + \\
& (A_4 + A_6) \left(r \frac{d^3 \theta_r}{dr^3} + 2 \frac{d^2 \theta_r}{dr^2} \right) - (A_5 + A_6) \left(\frac{1}{r} \frac{d\theta_r}{dr} \right. \\
& - \left. \frac{\theta_r}{r} \right) + g r + r \frac{d^2}{dr^2} [(M'_T)_r + (M''_T)_r] + 2 \frac{d}{dr} [\\
& (M'_T)_r + (M''_T)_r] - \frac{d}{dr} [(M'_T)_\alpha + (M''_T)_\alpha] = 0,
\end{aligned} \tag{2-27c}$$

where

$$A_1 = \frac{2Et}{1-\nu^2} + \frac{E_r \bar{t}}{1-\nu_{ra}\nu_{or}}, \tag{2-27d}$$

$$A_2 = \frac{2Et}{1-\nu^2} + \frac{E_\alpha \bar{t}}{1-\nu_{ra}\nu_{or}}, \tag{2-27e}$$

$$A_3 = \frac{2Et}{1+\nu} + \frac{E_r \bar{t}(1-\nu_{or})}{1-\nu_{ra}\nu_{or}}, \tag{2-27f}$$

$$A_4 = \frac{Et\bar{t}^2}{2(1-\nu^2)} + \frac{E_r \bar{t}^3}{12(1-\nu_{ra}\nu_{or})}, \tag{2-27g}$$

$$A_5 = \frac{E t \bar{t}^2}{2(1-\nu^2)} + \frac{E_{\theta} \bar{t}^3}{12(1-\nu_{\theta\nu_{\theta r})}} \quad (2-27h)$$

$$A_6 = \frac{E \bar{t} t^2}{2(1-\nu^2)} \quad (2-27i)$$

$$A_7 = G_i \bar{t} \quad (2-27j)$$

$$A_8 = \frac{2Et^3}{3(1-\nu^2)} + \frac{E \bar{t} t^2}{2(1-\nu^2)} \quad (2-27k)$$

b. Boundary Conditions in Terms of the Physical Displacements

When we use Equations (2-25) and (2-26), the boundary conditions, Equations (2-24), can also be expressed in terms of the physical displacements as

$$\left\{ A_1 \left[\frac{d\bar{u}_r}{dr} + \frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 \right] + (A_1 - A_3) \frac{\bar{u}_r}{r} + T_r \right\} \delta \bar{u}_r = 0 \quad (2-28a)$$

$$\left\{ A_4 \frac{d\theta_r}{dr} - A_6 \left(\frac{d^2 W}{dr^2} + \frac{\nu}{r} \frac{dW}{dr} \right) + A_9 \frac{1}{r} \theta_r + (\hat{M}_T)_r \right\} \delta \theta_r = 0 \quad (2-28b)$$

$$\left\{ \left\{ A_1 \left[\frac{d\bar{u}_r}{dr} + \frac{dW}{dr} \frac{dZ}{dr} + \frac{1}{2} \left(\frac{dW}{dr} \right)^2 + \frac{\bar{u}_r}{r} \right] - A_3 \frac{\bar{u}_r}{r} + T_r \right\} \left(\frac{dW}{dr} + \frac{dZ}{dr} \right) + \frac{d}{dr} \left[A_6 \left(\frac{d\theta_r}{dr} + \nu \frac{\theta_r}{r} \right) - (A_8 - A_6) \left(\frac{d^2 W}{dr^2} + \frac{\nu}{r} \frac{dW}{dr} \right) + (M'_T)_r + (M''_T)_r \right] + \frac{1-\nu}{r} \left[A_6 \left(\frac{d\theta_r}{dr} - \frac{\theta_r}{r} \right) - \right. \right.$$

$$-(A_8 - A_6) \left(\frac{d^2 W}{dr^2} - \frac{1}{r} \frac{dW}{dr} \right) - A_7 \left(\theta_r + \frac{dW}{dr} \right) + \frac{1}{r} \left[(M_T')_r + (M_T'')_r - (M_T')_\theta - (M_T'')_\theta \right] \delta W = 0 \quad (2-28c)$$

$$\left[A_6 \left(\frac{d\theta_r}{dr} + \frac{\nu}{r} \theta_r \right) - (A_8 - A_6) \left(\frac{d^2 W}{dr^2} + \frac{\nu}{r} \frac{dW}{dr} \right) + (M_T')_r + (M_T'')_r \right] \delta \left(\frac{dW}{dr} \right) = 0, \quad (2-28d)$$

where

$$A_9 = \frac{E \nu t \bar{t}^2}{2(1-\nu^2)} + \frac{E_r \nu_{or} \bar{t}^3}{12(1-\nu_{or}\nu_r)} \quad (2-28e)$$

The conditions at the center, $r = 0$, due to the axisymmetrical nature of the problem, are as follows

$$\bar{u}_r = 0, \quad (2-29a)$$

$$\theta_r = 0. \quad (2-29b)$$

$$\frac{dW}{dr} = 0, \quad (2-29c)$$

and

$$w \text{ is finite at } r = 0. \quad (2-29d)$$

The boundary conditions, Equations (2-28), together with the conditions at the center, Equations (2-29), and the equilibrium Equations (2-27) form a complete specification of the engineering problem.

CHAPTER III

NUMERICAL APPLICATIONS

In this chapter an axisymmetrical sandwich shallow shell of paraboloid geometry, $z = \frac{r_0^2}{2R_0} (1 - \frac{r^2}{r_0^2})$, is considered as an example for a numerical solution. The structure is under uniformly distributed mechanical loading as shown in Figure 4. The example does not include thermal effects. However, the method employed here is valid no matter whether the thermal effect is considered or not when the temperature distribution is assumed as a known function. Then one simply adds some terms due to the temperature distribution to the equilibrium equations, as well as to the boundary conditions. Furthermore, we consider the edge of the shell as clamped. The paraboloid geometrical configuration is a good approximation to a shallow spherical case.

1. Dimensionless Parameters

The equilibrium equations and the boundary conditions obtained in the previous chapters are in what is considered a "dimensional" form. Each of the elastic constants, E , ν , E_r , E_θ , $\nu_{r\theta}$, geometrical properties of the

structure, t , $\bar{t}$, r_0 , R_0 , and the space variable r have the appropriate physical units. It is customarily desirable to transform the differential equations as well as the boundary conditions into a "dimensionless" form. By solving the "dimensionless" form, one has the solution for a number of problems for a certain range of values of the parameters, and consequently a greater amount of information and more general applicability will be reached.

Let us introduce a set of dimensionless parameters

$$\eta = \frac{r}{r_0}, \quad 0 \leq \eta \leq 1, \quad (3-1a)$$

$$N = \frac{W}{h}, \quad (3-1b)$$

$$U = \frac{\bar{U}_r R_0}{h r_0}, \quad (3-1c)$$

$$\theta = \frac{r_0}{h} \theta_r, \quad (3-1d)$$

$$K = \frac{r_0^2}{2 R_0 h}. \quad (3-1e)$$

A. Equilibrium Equations in Dimensionless Form

By performing the transformation and dropping the temperature terms, Equation (2-27) becomes

$$B_1 \left[\gamma \frac{d^2 u}{d\gamma^2} + \frac{du}{d\gamma} - (\gamma^2 \frac{d^2 W}{d\gamma^2} + \gamma \frac{dW}{d\gamma}) + \frac{1}{2K} \gamma \frac{dW}{d\gamma} \frac{d^2 W}{d\gamma^2} \right] - B_2 \frac{u}{\gamma} + B_3 \left[\frac{1}{4K} \left(\frac{dW}{d\gamma} \right)^2 - \gamma \frac{dW}{d\gamma} \right] = 0, \quad (3-2a)$$

$$B_4 \left(\gamma \frac{d^2 \theta}{d\gamma^2} + \frac{d\theta}{d\gamma} \right) - B_5 \frac{\theta}{\gamma} - B_6 \left(\gamma \frac{d^3 W}{d\gamma^3} + \frac{d^2 W}{d\gamma^2} - \frac{1}{\gamma} \frac{dW}{d\gamma} \right) - B_7 (\gamma \theta + \gamma \frac{dW}{d\gamma}) = 0, \quad (3-2b)$$

$$\begin{aligned} & \left\{ B_1 K \left[\gamma \frac{dU}{d\gamma} - \gamma^2 \frac{dW}{d\gamma} + \frac{\gamma}{4K} \left(\frac{dW}{d\gamma} \right)^2 + U \right] - B_3 K U \right\} \frac{d^2 W}{d\gamma^2} + \\ & + \left\{ (B_1 - B_3) K \left[\frac{dU}{d\gamma} - \gamma \frac{dW}{d\gamma} + \frac{1}{4K} \left(\frac{dW}{d\gamma} \right)^2 \right] + B_2 K \frac{U}{\gamma} \right\} \frac{dW}{d\gamma} - \\ & - (4B_1 - 2B_3) K^2 \left[\gamma \frac{dU}{d\gamma} - \gamma^2 \frac{dW}{d\gamma} + \frac{1}{4K} \gamma \left(\frac{dW}{d\gamma} \right)^2 \right] - \\ & - 2(B_1 + B_2 - B_3) K^2 U - \left(\gamma \frac{d^4 W}{d\gamma^4} + 2 \frac{d^3 W}{d\gamma^3} - \frac{1}{\gamma} \frac{d^2 W}{d\gamma^2} + \right. \\ & + \left. \frac{1}{\gamma^2} \frac{dW}{d\gamma} \right) + (B_4 + B_6) \left(\gamma \frac{d^3 \theta}{d\gamma^3} + 2 \frac{d^2 \theta}{d\gamma^2} \right) - (B_5 + \\ & + B_6) \left(\frac{1}{\gamma} \frac{d\theta}{d\gamma} - \frac{\theta}{\gamma^2} \right) + \bar{g} \gamma = 0, \end{aligned} \quad (3-2c)$$

where

$$B_1 = \frac{2A_1 h^2}{A_8}, \quad (3-3a)$$

$$B_2 = \frac{2A_2 h^2}{A_8}, \quad (3-3b)$$

$$B_3 = \frac{2A_3 h^2}{A_8}, \quad (3-3c)$$

$$B_4 = \frac{A_4}{A_8} , \quad (3-3d)$$

$$B_5 = \frac{A_5}{A_8} , \quad (3-3e)$$

$$B_6 = \frac{A_6}{A_8} , \quad (3-3f)$$

$$B_7 = \frac{A_7 r_0^2}{A_8} , \quad (3-3g)$$

$$\bar{f} = \frac{f r_0^4}{A_8 h} . \quad (3-3h)$$

B. Boundary Conditions in Dimensionless Form

For a clamped edge, the boundary conditions, Equations (2-28a), (2-28b) and (2-28c), are automatically satisfied. Equation (2-28d) becomes, at $\eta = 1$,

$$B_6 \frac{d\theta}{d\eta} - (1 - B_6) \left(\frac{d^2 W}{d\eta^2} + \frac{\nu}{\eta} \frac{dW}{d\eta} \right) = 0 , \quad (3-4a)$$

while

$$\theta = u = \frac{dW}{d\eta} = 0 \quad \text{at } \eta = 0 , \quad (3-4b)$$

$$\theta = u = W = 0 \quad \text{at } \eta = 1 . \quad (3-4c)$$

2. Numerical Solution

It is obviously very difficult to solve exactly three coupled nonlinear differential equations. Accordingly, a certain approximate solution has been sought. A perturbation method with the center deflection in dimensionless form as the perturbing parameter is used to reduce the nonlinear differential equations into several sets of linear differential equations with variable coefficients. In solving each set of linear differential equations, the power-series expansion method is employed. A general recurrence formula is thus obtained from the differential equations and all the coefficients for the series can be determined by imposing the boundary conditions.

For a practical reason a finite number of terms of the power-series is assumed as an approximate solution. By using the recurrence formulae which are obtained from the differential equations and the boundary conditions, one can have a set of linear algebraic equations with undefined coefficients as unknowns. In solving these algebraic equations, a digital computer program of the matrix inversion written in Fortran II has been used. The computed results are plotted in Figure 5.

The problem is then reduced to an ordinary shallow shell by taking the thickness of the core equal to zero;

i.e., $\bar{t} = 0$. By doing so, one merely has two facings bonded together working as an ordinary shell structure of thickness $h = 2t$. The resulting curves are compared with known solutions (23, 24) as shown in Figure 6.

A. Sandwich Shell Structure

For an isotropic material we have $B_4 = B_5$,

$$P_1 = B_2, \bar{B}_1 = -\bar{B}_2 = (2B_3 - 4B_1)k^2 \text{ and } \bar{B}_4 = \bar{B}_3 = B_4 + B_6.$$

Therefore, Equations (3-2) can be rewritten in the form

$$B_1 \left[\gamma \frac{d^2 U}{d\gamma^2} + \frac{dU}{d\gamma} - \frac{U}{\gamma} - \left(\gamma^2 \frac{d^2 W}{d\gamma^2} + \gamma \frac{dW}{d\gamma} \right) + \frac{1}{2K} \gamma \frac{dW}{d\gamma} \frac{d^2 W}{d\gamma^2} \right] + B_3 \left[\frac{1}{4K} \left(\frac{dW}{d\gamma} \right)^2 - \gamma \frac{dW}{d\gamma} \right] = 0, \quad (3-5a)$$

$$B_4 \left(\gamma \frac{d^2 \theta}{d\gamma^2} + \frac{d\theta}{d\gamma} - \frac{\theta}{\gamma} \right) - B_6 \left(\gamma \frac{d^3 W}{d\gamma^3} + \frac{d^2 W}{d\gamma^2} - \frac{1}{\gamma} \frac{dW}{d\gamma} \right) - B_7 \left(\gamma \theta + \gamma \frac{dW}{d\gamma} \right) = 0, \quad (3-5b)$$

$$\begin{aligned} & \left\{ B_1 K \left[\gamma \frac{dU}{d\gamma} - \gamma^2 \frac{dW}{d\gamma} + \frac{\gamma}{4K} \left(\frac{dW}{d\gamma} \right)^2 + U \right] - B_3 K U \right. \\ & \left. \right\} \frac{d^2 W}{d\gamma^2} + \left\{ (B_1 - B_3) K \left[\frac{dU}{d\gamma} - \gamma \frac{dW}{d\gamma} + \frac{1}{4K} \left(\frac{dW}{d\gamma} \right)^2 \right] + \right. \\ & \left. + B_1 K \frac{U}{\gamma} \right\} \frac{dW}{d\gamma} + \bar{B}_1 \left[\gamma \frac{dU}{d\gamma} - \gamma^2 \frac{dW}{d\gamma} + \frac{1}{4K} \gamma \left(\frac{dW}{d\gamma} \right)^2 + U \right] - \left(\gamma \frac{d^4 W}{d\gamma^4} + 2 \frac{d^3 W}{d\gamma^3} - \frac{1}{\gamma} \frac{d^2 W}{d\gamma^2} + \frac{1}{\gamma^2} \frac{dW}{d\gamma} \right) + \bar{B}_3 \left(\right. \\ & \left. \gamma \frac{d^3 \theta}{d\gamma^3} + 2 \frac{d^2 \theta}{d\gamma^2} - \frac{1}{\gamma} \frac{d\theta}{d\gamma} + \frac{\theta}{\gamma^2} \right) + \bar{B}_7 \gamma = 0. \end{aligned} \quad (3-5c)$$

Let the functions W , u , θ and $\bar{q}$ be expanded in the following form

$$W = W_1(\gamma) w_0 + W_2(\gamma) w_0^2 + \dots, \quad (3-6a)$$

$$u = U_1(\gamma) w_0 + U_2(\gamma) w_0^2 + \dots, \quad (3-6b)$$

$$\theta = \theta_1(\gamma) w_0 + \theta_2(\gamma) w_0^2 + \dots, \quad (3-6c)$$

$$\bar{q} = q_1 w_0 + q_2 w_0^2 + \dots. \quad (3-6d)$$

Substituting Equation (3-6) into Equation (3-5) and equating similar powers of w_0 result in a sequence of sets of simultaneous equations for $W_n(\gamma)$, $U_n(\gamma)$, $\theta_n(\gamma)$ and q_n . Considering on expansion of two terms, we have two sets of simultaneous equations. From the first set,

$$B_1 \left[\gamma \frac{d^2 U_1}{d\gamma^2} + \frac{dU_1}{d\gamma} - \frac{U_1}{\gamma} - \left(\gamma^2 \frac{d^2 W_1}{d\gamma^2} + \gamma \frac{dW_1}{d\gamma} \right) \right] - B_3 \gamma \frac{dW_1}{d\gamma} = 0, \quad (3-7a)$$

$$B_4 \left[\gamma^2 \frac{d^2 \theta_1}{d\gamma^2} + \frac{d\theta_1}{d\gamma} - \frac{\theta_1}{\gamma} \right] - B_6 \left(\gamma \frac{d^3 W_1}{d\gamma^3} + \frac{d^2 W_1}{d\gamma^2} - \frac{1}{\gamma} \frac{dW_1}{d\gamma} \right) - B_7 \left(\gamma \theta_1 + \gamma \frac{dW_1}{d\gamma} \right) = 0, \quad (3-7b)$$

$$\begin{aligned} \bar{B}_1 \left(\gamma \frac{dU_1}{d\gamma} - \gamma^2 \frac{dW_1}{d\gamma} + U_1 \right) - \left(\gamma \frac{d^4 W_1}{d\gamma^4} + 2 \frac{d^3 W_1}{d\gamma^3} - \frac{1}{\gamma} \frac{d^2 W_1}{d\gamma^2} + \frac{1}{\gamma^2} \frac{dW_1}{d\gamma} \right) + \bar{B}_3 \left(\gamma \frac{d^3 \theta_1}{d\gamma^3} + 2 \frac{d^2 \theta_1}{d\gamma^2} - \right. \end{aligned}$$

$$-\frac{1}{\eta} \frac{d\theta_1}{d\eta} + \frac{\theta_1}{\eta^2} + f_1 \eta = 0. \quad (3-7c)$$

The corresponding boundary conditions are

$$\theta_1 = U_1 = \frac{dW_1}{d\eta} = 0, \quad (3-7d)$$

$$W_1 = 1, \quad \text{at } \eta = 0,$$

and

$$\theta_1 = U_1 = W_1 = 0 \quad (3-7e)$$

$$B_6 \frac{d\theta_1}{d\eta} - (1 - B_6) \left(\frac{d^2 W_1}{d\eta^2} + \frac{\lambda}{\eta} \frac{dW_1}{d\eta} \right) = 0, \quad \text{at } \eta = 1.$$

The second set of differential equations are

$$B_1 \left[\eta \frac{d^2 U_2}{d\eta^2} + \frac{dU_2}{d\eta} - \frac{U_2}{\eta} - \left(\eta^2 \frac{d^2 W_2}{d\eta^2} + \eta \frac{dW_2}{d\eta} \right) \right] - \\ - B_3 \eta \frac{dW_2}{d\eta} + \frac{B_1}{2K} \eta \left(\frac{dW_1}{d\eta} \right) \left(\frac{d^2 W_1}{d\eta^2} \right) + \frac{B_3}{4K} \left(\frac{dW_1}{d\eta} \right)^2 = 0, \quad (3-8a)$$

$$B_4 \left(\eta \frac{d^2 \theta_2}{d\eta^2} + \frac{d\theta_2}{d\eta} - \frac{\theta_2}{\eta} \right) - B_6 \left(\eta \frac{d^3 W_2}{d\eta^3} + \frac{d^2 W_2}{d\eta^2} - \right. \\ \left. - \frac{1}{\eta} \frac{dW_2}{d\eta} \right) - B_7 (\eta \theta_2 + \eta \frac{dW_1}{d\eta}) = 0, \quad (3-8b)$$

$$\left\{ B_1 K \left(\eta \frac{dU_1}{d\eta} - \eta^2 \frac{dW_1}{d\eta} + U_1 \right) - B_3 K U_1 \right\} \frac{d^2 W_1}{d\eta^2} + \\ + \left\{ (B_1 - B_3) K \left(\frac{dU_1}{d\eta} - \eta \frac{dW_1}{d\eta} \right) + B_1 K \frac{U_1}{\eta} \right\} \frac{dW_1}{d\eta} +$$

$$\begin{aligned}
& + \bar{B}_1 \left[\eta \frac{dU_2}{d\eta} + U_2 - \eta^2 \frac{dW_2}{d\eta} + \frac{1}{4K} \eta \left(\frac{dW_2}{d\eta} \right)^2 \right] - \\
& - \left(\eta \frac{d^4 W_2}{d\eta^4} + 2 \frac{d^3 W_2}{d\eta^3} - \frac{1}{\eta} \frac{d^2 W_2}{d\eta^2} + \frac{1}{\eta^2} \frac{dW_2}{d\eta} \right) + \\
& + \bar{B}_3 \left(\eta \frac{d^3 \theta_2}{d\eta^3} + 2 \frac{d^2 \theta_2}{d\eta^2} - \frac{1}{\eta} \frac{d\theta_2}{d\eta} + \frac{\theta_2}{\eta} \right) + g_2 \eta = 0.
\end{aligned} \quad (3-8c)$$

The corresponding boundary conditions are

$$\begin{aligned}
U_2 = \theta_2 = \frac{dW_2}{d\eta} = W_2 = 0 \\
\text{at } \eta = 0,
\end{aligned} \quad (3-8d)$$

and

$$\begin{aligned}
U_2 = \theta_2 = W_2 = 0, \\
B_6 \frac{d\theta_2}{d\eta} - (1 - B_6) \left(\frac{d^2 W_2}{d\eta^2} + \frac{1}{\eta} \frac{dW_2}{d\eta} \right) = 0 \\
\text{at } \eta = 1.
\end{aligned} \quad (3-8e)$$

Let us solve the first set of differential equations. From Equation (3-7a), we have

$$\frac{d}{d\eta} (\eta U_1) - \eta^2 \frac{dW_1}{d\eta} = \frac{B_3}{B_1} \eta W_1 + c_1 \eta, \quad (3-9)$$

where c_1 is an integration constant. By substituting Equation (3-9) into Equation (3-7c) and performing an integration

with respect to η once, we have

$$\begin{aligned} \bar{B}_3 \left(\eta \frac{d^2 \theta_1}{d\eta^2} + \frac{d\theta_1}{d\eta} - \frac{\theta_1}{\eta} \right) &= \eta \frac{d^3 W_1}{d\eta^3} + \frac{d^2 W_1}{d\eta^2} - \frac{1}{\eta} \frac{dW_1}{d\eta} - \\ &- C_2 - \frac{\bar{B}_1 B_2}{B_1} \int \eta W_1 d\eta - \frac{\bar{B}_1 C_1}{2} \eta^2 + \frac{1}{2} g_1 \eta^2. \end{aligned} \quad (3-10)$$

Substituting Equation (3-10) into Equation (3-8b), and solving for θ_1 , we obtain

$$\begin{aligned} \theta_1 &= \frac{1}{B_7} \left(\frac{B_4}{B_3} - B_6 \right) \left(\frac{d^3 W_1}{d\eta^3} + \frac{1}{\eta} \frac{d^2 W_1}{d\eta^2} - \frac{1}{\eta^2} \frac{dW_1}{d\eta} \right) - \\ &- \frac{B_4 C_2}{\bar{B}_3 B_7} \frac{1}{\eta} - \frac{\bar{B}_1 B_2 B_4}{B_1 \bar{B}_3 B_7} \frac{1}{\eta} \int \eta W_1 d\eta - \frac{\bar{B}_1 B_4}{2 \bar{B}_3 B_7} C_1 \eta - \\ &- \frac{B_4 g_1}{2 \bar{B}_3 B_7} \eta - \frac{dW_1}{d\eta}. \end{aligned} \quad (3-11)$$

Let the function W_1 be expressed in a power series

$$W_1 = 1 + \sum_{p=1}^{\infty} \alpha_{2p} \eta^{2p}. \quad (3-12)$$

By substituting Equation (3-12) into Equation (3-11) we obtain

$$\begin{aligned} \theta_1 &= -\frac{B_4 C_2}{\bar{B}_3 B_7} \frac{1}{\eta} + \left[32 \left(\frac{B_4}{\bar{B}_3 B_7} - \frac{B_6}{B_7} \right) \alpha_4 - \frac{\bar{B}_1 B_2 B_4}{2 B_1 \bar{B}_3 B_7} - \right. \\ &- \frac{\bar{B}_1 B_4}{2 \bar{B}_3 B_7} C_1 - \frac{B_4 g_1}{2 \bar{B}_3 B_7} - 2 \alpha_2 \left. \right] \eta + \sum_{p=1}^{\infty} \left[\left(\frac{B_4}{\bar{B}_3 B_7} - \right. \right. \\ &- \frac{B_6}{B_7} \left. \right) 8(p+1)(p+2) \alpha_{2p+2} - \frac{\bar{B}_1 B_2 B_4}{B_1 \bar{B}_3 B_7} \frac{\alpha_{2p}}{2(p+1)} - \\ &- 2(p+1) \alpha_{2p+2} \left. \right] \eta^{2p+1}. \end{aligned} \quad (3-13a)$$

By applying the boundary condition (3-7d), we have $c_2 = 0$,
and

$$\theta_i = \sum_{p=0}^{\infty} \gamma_{2p+1} \gamma^{2p+1}, \quad (3-13b)$$

where

$$\begin{aligned} \gamma_i = 32 \left(\frac{B_4}{B_3 B_7} - \frac{B_6}{B_7} \right) \alpha_4 - \frac{\bar{B}_1 B_2 B_4}{2 B_1 \bar{B}_3 B_7} - \frac{\bar{B}_1 B_4}{2 \bar{B}_3 B_7} C_1 - \\ - \frac{B_4 q_1}{2 \bar{B}_3 B_7} - 2 \alpha_2, \end{aligned} \quad (3-14a)$$

and

$$\begin{aligned} \gamma_{2p+1} = 8 \left(\frac{B_4}{\bar{B}_1 B_7} - \frac{B_6}{B_7} \right) (p+1)(p+2) \alpha_{2p+4} \\ - \frac{\bar{B}_1 B_2 B_4}{2 B_1 \bar{B}_3 B_7} \frac{\alpha_{2p}}{p+1} - 2(p+1) \alpha_{2p+2}. \end{aligned} \quad (3-14b)$$

$p = 1, 2, 3, \dots$

Also, from Equation (3-9), we obtain

$$U_i = \sum_{p=0}^{\infty} \beta_{2p+1} \gamma^{2p+1}, \quad (3-15)$$

where

$$\beta_i = \frac{B_2}{2 B_1} + \frac{C_1}{2}, \quad (3-16a)$$

and

$$\begin{aligned} \beta_{2p+1} = \left[\frac{p}{p+1} + \frac{B_3}{2(p+1) B_1} \right] \alpha_{2p}, \\ p = 1, 2, 3, \dots \end{aligned} \quad (3-16b)$$

Substituting Equations (3-12) and (3-13a) into Equation (3-10) and equating the coefficients of the same order terms we obtain the recurrence formula for the α 's as

$$\frac{\bar{B}_1 B_3}{2 B_1} + \frac{\bar{B}_1 C_1}{2} - 32(1 + \bar{B}_3) \alpha_4 + 1152 \left(\frac{B_4}{B_7} - \frac{B_6 \bar{B}_3}{B_7} \right) \alpha_6 - 2 \frac{\bar{B}_1 B_2 B_3}{B_1 B_7} \alpha_2 + \frac{q_1}{2} = 0, \quad (3-17)$$

and

$$\begin{aligned} & 64 \left(\frac{B_4}{B_7} - \frac{B_6 \bar{B}_3}{B_7} \right) (p+2)^2 (p+1)^2 p^2 \alpha_{2p+4} - \\ & 16 (1 + \bar{B}_3) (p+1)^2 p^2 \alpha_{2p+2} - 4 \frac{\bar{B}_1 B_2 B_4}{B_1 B_7} p^2 \alpha_{2p} + \\ & + \frac{\bar{B}_1 B_3}{B_1} \alpha_{2p-2} = 0. \quad p = 2, 3, 4, \dots \end{aligned} \quad (3-18)$$

By this recurrence formula (3-18) it can be proven that the power series is convergent (Appendix A). We have only four unknowns α_2 , α_4 , c_1 and q_1 to be solved from four boundary conditions, Equation (3-7e).

As a rough approximation, let us assume

$$W_i = 1 + \alpha_2 \eta^2 + \alpha_4 \eta^4 + \alpha_6 \eta^6, \quad (3-19a)$$

$$U_i = \beta_1 \eta + \beta_3 \eta^3 + \beta_5 \eta^5 + \beta_7 \eta^7, \quad (3-19b)$$

$$\theta_i = \gamma_1 \eta + \gamma_3 \eta^3 + \gamma_5 \eta^5 + \gamma_7 \eta^7. \quad (3-19c)$$

By using the recurrence formulae we have a set of algebraic equations

$$\frac{q_1}{2} - 2b_2\alpha_2 - 32(1+\bar{B}_3)\alpha_4 + 1152b_1\alpha_6 + \frac{\bar{B}_1}{2}c_1 = -\frac{\bar{B}_1B_3}{2B_1}, \quad (3-20a)$$

$$\alpha_2 + \alpha_4 + \alpha_6 = -1, \quad (3-20b)$$

$$\bar{B}_3\gamma_3 + 4\bar{B}_3\alpha_4 - 14 + b_1\alpha_6 + \frac{b_1}{4}\alpha_2 = 0, \quad (3-20c)$$

$$B_6\gamma_1 + 3B_6\gamma_3 + 5B_6\gamma_5 + 7B_6\gamma_7 - (1-B_6)[2(1+\gamma)\alpha_2 + 4(3+\gamma)\alpha_4 + 6(5+\gamma)\alpha_6] = 0, \quad (3-20d)$$

$$\beta_1 + \beta_3 + \beta_5 + \beta_7 = 0, \quad (3-20e)$$

$$\beta_3 - \left(\frac{1}{2} + \frac{B_3}{4B_1}\right)\alpha_2 = 0, \quad (3-20f)$$

$$\beta_5 - \left(\frac{2}{3} + \frac{B_5}{6B_1}\right)\alpha_4 = 0, \quad (3-20g)$$

$$\beta_7 - \left(\frac{3}{4} + \frac{B_7}{8B_1}\right)\alpha_6 = 0, \quad (3-20h)$$

$$\beta_1 - \frac{c_1}{2} = \frac{B_3}{2B_1}, \quad (3-20i)$$

$$\gamma_1 - \frac{32b_1}{B_3}\alpha_4 + \frac{\bar{B}_1B_4}{2\bar{B}_3B_7}c_1 + \frac{B_4}{2\bar{B}_3B_7}q_1 + 2\alpha_2 = -\frac{b_2}{2\bar{B}_3}, \quad (3-20j)$$

$$\gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 = 0, \quad (3-20k)$$

$$\gamma_5 + \frac{b_2}{6\bar{B}_3} \alpha_4 + 6\alpha_6 = 0, \quad (3-20l)$$

$$\gamma_7 + \frac{b_2}{8\bar{B}_3} \alpha_6 = 0, \quad (3-20m)$$

where

$$\begin{aligned} b_1 &= \frac{B_4}{B_7} - \frac{B_6 \bar{B}_3}{B_7}, \\ b_2 &= \frac{\bar{B}_1 B_3 B_4}{B_1 B_7}, \\ b_3 &= \frac{\bar{B}_1 B_3}{B_1}. \end{aligned} \quad (3-20n)$$

These algebraic equations (3-20) are solved by the first part of a digital computer program in Fortran II (Appendix B). The results are to be used for the second set of the differential equations.

The same procedure will be applied in solving the second set of the differential equations in which the non-linear terms of the W_1 and U_1 are treated as known functions of η . From Equation (3-8a) we have

$$\begin{aligned} B_1 \left[\eta \frac{d^2 U_2}{d\eta^2} + \frac{dU_2}{d\eta} - \frac{U_2}{\eta} - \left(\eta^2 \frac{d^2 W_2}{d\eta^2} + \eta \frac{dW_2}{d\eta} \right) \right] - \\ - B_3 \eta \frac{dW_2}{d\eta} + \chi_1(\eta) = 0, \end{aligned} \quad (3-21a)$$

where

$$\chi_1(\eta) = \sum_{p=1}^{\infty} \sum_{s=1}^{\infty} \left[\frac{B_1}{2K} (2s-1) + \frac{\bar{B}_2}{4K} \right] 4ps \alpha_{2p} \alpha_{2s} \eta^{2(p+s)-2}. \quad (3-21b)$$

By integration of Equation (3-21a) with respect to η , we obtain

$$\frac{d}{d\eta}(\eta U_2) = \eta^2 \frac{dW_2}{d\eta} + \frac{B_2}{B_1} \eta W_2 - \frac{\eta}{B_1} \int \frac{1}{\eta} \chi_1(\eta) d\eta + \bar{C}_1 \eta, \quad (3-22)$$

where $\bar{C}_1$ is an integration constant.

Substituting Equation (3-22) into (3-8c) and integrating once with respect to η , we obtain

$$\begin{aligned} \eta \frac{d^2 \theta_2}{d\eta^2} + \frac{d\theta_2}{d\eta} - \frac{\theta_2}{\eta} &= \frac{1}{B_3} \left(\eta \frac{d^3 W_2}{d\eta^3} + \frac{d^2 W_2}{d\eta^2} - \frac{1}{\eta} \frac{dW_2}{d\eta} \right) - \\ &\quad - \frac{\bar{C}_2}{B_3} - \frac{1}{B_3} \int \gamma_1(\eta) d\eta - \frac{\bar{B}_1 B_2}{B_1 B_3} \int \eta W_2 d\eta + \\ &\quad + \frac{\bar{B}_1}{B_1 B_3} \int \eta \int \frac{1}{\eta} \chi(\eta) d\eta d\eta - \frac{\bar{B}_1 \bar{C}_1}{2 B_3} \eta^2 - \\ &\quad - \frac{\bar{C}_2}{2 B_3} \eta^2, \end{aligned} \quad (3-23)$$

where $\bar{C}_2$ is an integration constant,

and

$$\begin{aligned} \gamma_1(\eta) &= \sum_{s=1}^{\infty} 4(2B_1 - B_3) K S^2 \beta_s \alpha_{2s} \eta^{2s-1} + \sum_{p=2}^{\infty} \sum_{s=1}^{\infty} \left\{ 4SK \left[\right. \right. \\ &\quad \left. \left. 2SpB_1 - (p+s-1)B_3 \right] \alpha_{2s} \beta_{2p-1} - 2KS(p-1) \left[\right. \right. \\ &\quad \left. \left. 2(2s+1)B_1 - 3B_3 \right] \alpha_{2s} \alpha_{2p-2} \right\} \eta^{2(p+s)-3}. \end{aligned} \quad (3-24)$$

Substituting Equation (3-23) into Equation (3-8b) and solving for θ_2 , we have

$$\begin{aligned}
\theta_2 = & \frac{b_1}{B_3} \left(\frac{d^3 W_2}{d\gamma^3} + \frac{1}{\gamma} \frac{d^2 W_2}{d\gamma^2} - \frac{1}{\gamma^2} \frac{dW_2}{d\gamma} \right) - \\
& - \frac{B_4}{B_1 B_3} \frac{1}{\gamma} \int y_1(\gamma) d\gamma - \frac{b_2}{B_3} \frac{1}{\gamma} \int \gamma W_2 d\gamma + \\
& + \frac{\bar{B}_1 B_4}{B_1 \bar{B}_3 B_7} \frac{1}{\gamma} \int \gamma \int \frac{1}{\gamma} \chi_1(\gamma) d\gamma d\gamma - \frac{\bar{B}_1 B_4}{2 \bar{B}_3 B_7} \bar{C}_1 \gamma - \\
& - \frac{B_4}{2 B_1 \bar{B}_3} g_2 \gamma - \frac{dW_2}{d\gamma} - \frac{\bar{C}_2}{B_3 \gamma} .
\end{aligned} \tag{3-25}$$

Let W_2 be expressed in a power-series

$$W_2 = \sum_{p=1}^{\infty} \bar{\alpha}_{2p} \gamma^{2p} . \tag{3-26}$$

Substituting Equation (3-26) into Equation (3-25),

$$\theta_2 = \sum_{p=1}^{\infty} \bar{\gamma}_{2p-1} \gamma^{2p-1} , \tag{3-27a}$$

where

$$\begin{aligned}
\bar{\gamma}_1 = & \frac{32 b_1}{B_3} \bar{\alpha}_4 - \frac{B_4}{B_1 B_3} (4 B_1 - 2 B_3) K \beta_1 \alpha_2 - \\
& - \frac{\bar{B}_1 B_4}{2 \bar{B}_3 B_7} \bar{C}_1 - \frac{B_4}{2 \bar{B}_3 B_7} g_2 - 2 \bar{\alpha}_2 ,
\end{aligned} \tag{3-27b}$$

$$\begin{aligned}
\bar{\gamma}_{2p-1} = & \frac{b_1}{B_3} (2p+2)^2 (2p) \bar{\alpha}_{2p+2} - \frac{B_4}{B_1 B_3} \left\{ 2p(2B_1 - \right. \\
& - B_3) K \beta_1 \alpha_{2p} + \sum_{s=2}^p \frac{K}{p} \left\{ 2(p-s+1) [2(p-s+1) s B_1 - \right. \\
& - p B_3] \alpha_{2(p-s+1)} \beta_{2s-1} - (p-s+1)(s-1) [2(2p - \\
& - 2s+3) B_1 - 3 B_3] \alpha_{2(p-s+1)} \alpha_{2s-2} \} \} -
\end{aligned}$$

$$\begin{aligned}
& - \frac{b_2}{B_3} \frac{\bar{\alpha}_{2p-2}}{2^p} + \frac{\bar{B}_1 B_4}{B_1 \bar{B}_3 B_7 K} \sum_{s=1}^{p-1} \frac{S(p-s)}{p(p-1)} \left[\frac{B_1}{2} (2p - \right. \\
& \left. - 2S - 1) + \frac{B_3}{4} \right] \alpha_{2s} \alpha_{2(p-s)} - 2p \bar{\alpha}_{2p} , \quad (3-27c)
\end{aligned}$$

for $p = 2, 3, 4, \dots$

and $\bar{\alpha}_2 = 0$, by the boundary condition (3-8d).

Substituting Equation (3-26) into Equation (3-22) and solving for U_2 ,

$$U_2 = \sum_{p=0}^{\infty} \bar{\beta}_{2p+1} \eta^{2p+1}, \quad (3-28a)$$

where

$$\bar{\beta}_1 = \frac{\bar{C}_1}{2}, \quad (3-28b)$$

and

$$\begin{aligned}
\bar{\beta}_{2p+1} = & \left[\frac{p}{p+1} + \frac{B_3}{2(p+1)B_1} \right] \bar{\alpha}_{2p} - \frac{1}{4KB_1} \sum_{s=1}^p \frac{S(p-s+1)}{p(p+1)} \times \\
& [2B_1(2p-2S+1) + B_3] \alpha_{2s} \alpha_{2(p-s+1)}, \quad (3-28c)
\end{aligned}$$

for $p = 1, 2, 3, \dots$

Finally by imposing Equations (3-26), (3-27) and (3-28) on Equation (2-23) and equating the coefficients for the same order terms, we have the recurrence formula for $\bar{\alpha}$'s.

$$\begin{aligned}
& 1152b_1 \bar{\alpha}_6 - 2b_2 \bar{\alpha}_2 - 32(\bar{B}_3 + 1) \bar{\alpha}_4 + \frac{\bar{B}_1 \bar{C}_1}{2} + \frac{g_2}{2} \\
& = \frac{4B_3 K}{B_7} \left[8(2B_1 - B_3) \beta_1 \alpha_4 + 4(2B_1 - B_3) \alpha_2 \beta_3 - 3(2B_1 - \right. \\
& \quad \left. - B_3) \alpha_2^2 \right] - 2(2B_1 - B_3) K \beta_1 \alpha_2 - \frac{4\bar{B}_1 B_3}{B_1 B_7} \left(\frac{B_1}{2K} + \frac{B_3}{4K} \right) \alpha_2^2, \quad (3-29)
\end{aligned}$$

$$\begin{aligned}
& 32b_1(\rho+2)^2(\rho+1)^2 \rho \bar{\alpha}_{2\rho+4} - 2b_2 \rho \bar{\alpha}_{2\rho} - 8(1 + \bar{B}_3) \times \\
& \rho(\rho+1)^2 \bar{\alpha}_{2\rho+2} + \frac{\bar{B}_1 B_3}{2B_1} \frac{1}{\rho} \bar{\alpha}_{2\rho-2} = \frac{B_4}{B_7} \left\{ 8(\rho+ \right. \\
& \quad \left. + 1)^2 \rho(2B_1 - B_3) K \beta_1 \alpha_{2\rho+2} + \sum_{s=2}^{\rho} 8K\rho \left\{ (\rho-s+2) \left[\right. \right. \right. \\
& \quad \left. \left. \left. 2(\rho-s+2) s B_1 - (\rho+1) B_3 \right] \alpha_{2(\rho-s+2)} \beta_{2s-1} - \right. \right. \\
& \quad \left. \left. - \rho(s-1) \left[2(2\rho-2s+5) B_1 - 3B_3 \right] \alpha_{2(\rho-s+2)} \right\} \right. \\
& \quad \left. \alpha_{2s-2} \right\} \left. \right\} - \left\{ 2\rho(2B_1 - B_3) K \beta_1 \alpha_{2\rho} + \sum_{s=2}^{\rho} \frac{K}{\rho} \left\{ 2(\right. \right. \\
& \quad \left. \left. \rho-s+1) \left[2s(\rho-s+1) B_1 - \rho B_3 \right] \alpha_{2(\rho-s+1)} \beta_{2s-1} - \right. \right. \\
& \quad \left. \left. - (\rho-s+1)(s-1) \left[2(2\rho-2s+3) B_1 - 3B_3 \right] \alpha_{2(\rho-s+1)} \right\} \right. \\
& \quad \left. \alpha_{2s-2} \right\} \left. \right\} - \frac{\bar{B}_1 B_4}{B_1 B_7 K} \sum_{s=1}^{\rho} s(\rho-s+1) \left[2(2\rho-2s+1) B_1 + \right. \\
& \quad \left. + B_3 \right] \alpha_{2s} \alpha_{2(\rho-s+1)} + \frac{\bar{B}_1}{4KB_1} \sum_{s=1}^{\rho-1} \frac{s(\rho-s)}{\rho(\rho-1)} \left[2(2\rho- \right.
\end{aligned}$$

$$-25-1)B_1 + B_3] \alpha_{25} \alpha_{2(p-5)} . \quad (3-30)$$

for $p = 2, 3, 4, \dots$

Here we have four unknowns $\bar{\alpha}_2$, $\bar{\alpha}_4$, $\bar{c}_1$ and q_2 to be determined from the four boundary conditions, Equation (3-8e).

We set approximately

$$W_2 = \bar{\alpha}_2 \gamma^2 + \bar{\alpha}_4 \gamma^4 + \bar{\alpha}_6 \gamma^6 + \bar{\alpha}_8 \gamma^8 + \bar{\alpha}_{10} \gamma^{10}, \quad (3-31a)$$

$$\theta_2 = \bar{\gamma}_1 \gamma + \bar{\gamma}_3 \gamma^3 + \bar{\gamma}_5 \gamma^5 + \bar{\gamma}_7 \gamma^7 + \bar{\gamma}_9 \gamma^9 + \bar{\gamma}_{11} \gamma^{11}, \quad (3-31b)$$

$$U_2 = \bar{\beta}_1 \gamma + \bar{\beta}_3 \gamma^3 + \bar{\beta}_5 \gamma^5 + \bar{\beta}_7 \gamma^7 + \bar{\beta}_9 \gamma^9 + \bar{\beta}_{11} \gamma^{11}. \quad (3-31c)$$

Correspondingly, we have a second set of algebraic equations to determine these coefficients

$$\begin{aligned} & \frac{f_2}{2} - 2b_2 \bar{\alpha}_2 - 32(1 + \bar{B}_3) \bar{\alpha}_4 + 1152 b_1 \bar{\alpha}_6 + \frac{\bar{B}_1}{2} \bar{c}_1 \\ &= \frac{4KB_4}{B_7} \left[8(2B_1 - B_3) \beta_1 \alpha_4 + 4(2B_1 - B_3) \alpha_2 \beta_3 - \right. \\ & \quad \left. - 3(2B_1 - B_3) \alpha_2^2 \right] - 2K(2B_1 - B_3) \beta_1 \alpha_2 - \\ & \quad - \frac{2\bar{B}_1 B_4}{B_1 B_7 K} \left(B_1 + \frac{B_4}{2} \right) \alpha_2^2, \end{aligned} \quad (3-32a)$$

$$\bar{\alpha}_2 + \bar{\alpha}_4 + \bar{\alpha}_6 + \bar{\alpha}_8 + \bar{\alpha}_{10} = 0, \quad (3-32b)$$

$$\begin{aligned} & \bar{B}_3 \bar{\gamma}_3 + 4 \bar{B}_3 \bar{\alpha}_4 - 144 b_1 \bar{\alpha}_6 + \frac{b_2}{4} \bar{\alpha}_2 \\ &= -\frac{B_4 K}{B_7} \left\{ 4(2B_1 - B_3) \beta_1 \alpha_4 + \frac{1}{2} \left[4(2B_1 - B_3) \alpha_2 \beta_3 - \right. \right. \\ & \quad \left. \left. - 3(2B_1 - B_3) \alpha_2^2 \right] \right\} + \frac{\bar{B}_1 B_4}{2B_1 B_7 K} \left(\frac{B_1}{2} + \frac{B_3}{4} \right) \alpha_2^2, \end{aligned} \quad (3-32c)$$

$$\begin{aligned} & B_6 [\bar{\gamma}_1 + 3\bar{\gamma}_3 + 5\bar{\gamma}_5 + 7\bar{\gamma}_7 + 9\bar{\gamma}_9 + 11\bar{\gamma}_{11}] - (1 - B_6) [\\ & 2(1+\nu) \bar{\alpha}_2 + 4(3+\nu) \bar{\alpha}_4 + 6(5+\nu) \bar{\alpha}_6 + 8(7+\nu) \bar{\alpha}_8 + 10(9+\nu) \bar{\alpha}_{10}] = 0, \end{aligned} \quad (3-32d)$$

$$\bar{\beta}_1 + \bar{\beta}_3 + \bar{\beta}_5 + \bar{\beta}_7 + \bar{\beta}_9 + \bar{\beta}_{11} = 0, \quad (3-32e)$$

$$\bar{\beta}_3 - \left(\frac{1}{2} + \frac{B_3}{4B_1} \right) \bar{\alpha}_2 = -\frac{1}{2B_1 K} \left(\frac{B_1}{2} + \frac{B_3}{4} \right) \alpha_2^2, \quad (3-32f)$$

$$\bar{\beta}_5 - \left(\frac{2}{3} + \frac{B_3}{6B_1} \right) \bar{\alpha}_4 = -\frac{1}{3B_1 K} \left(2B_1 + \frac{B_3}{2} \right) \alpha_2 \alpha_4, \quad (3-32g)$$

$$\begin{aligned} \bar{\beta}_7 - \left(\frac{3}{4} + \frac{B_3}{8B_1} \right) \bar{\alpha}_6 &= -\frac{1}{B_1 K} \left[\left(\frac{3}{4} B_1 + \frac{B_3}{8} \right) \alpha_2 \alpha_6 + \right. \\ & \quad \left. + \left(\frac{B_1}{2} + \frac{B_3}{12} \right) \alpha_4^2 \right], \end{aligned} \quad (3-32h)$$

$$\bar{\beta}_1 - \frac{\bar{C}_1}{2} = 0, \quad (3-32i)$$

$$\begin{aligned} \bar{\gamma}_1 &= \frac{32}{B_3} b_1 \bar{\alpha}_4 + \frac{\bar{B}_1 B_4}{2 \bar{B}_3 B_7} \bar{C}_1 + \frac{B_4}{2 \bar{B}_3 B_7} \bar{g}_2 + 2 \bar{\alpha}_2 \\ &= -\frac{2 B_4}{\bar{B}_3 B_7} (2 B_1 - B_3) K \beta_1 \alpha_2, \end{aligned} \quad (3-32j)$$

$$\bar{\gamma}_1 + \bar{\gamma}_3 + \bar{\gamma}_5 + \bar{\gamma}_7 + \bar{\gamma}_9 + \bar{\gamma}_{11} = 0, \quad (3-32k)$$

$$\begin{aligned} \bar{\gamma}_5 &= 384 \frac{b_1}{B_3} \bar{\alpha}_8 + \frac{b_2}{6 B_3} \bar{\alpha}_4 + 6 \bar{\alpha}_6 \\ &= -\frac{B_4 K}{\bar{B}_3 B_7} \left\{ 6(2 B_1 - B_3) \beta_1 \alpha_6 + \frac{1}{3} [4(8 B_1 - 3 B_3) \alpha_4 \beta_3 + \right. \\ &\quad \left. + 6(2 B_1 - B_3) \alpha_2 \beta_5 - 4(8 B_1 - 3 B_3) \alpha_2 \alpha_4] \right\} + \\ &\quad + \frac{\bar{B}_1 B_4}{3 B_1 \bar{B}_3 B_7 K} \left(2 B_1 + \frac{B_3}{2} \right) \alpha_2 \alpha_4, \end{aligned} \quad (3-32l)$$

$$\begin{aligned} \bar{\gamma}_7 &= 800 \frac{b_1}{B_3} \bar{\alpha}_{10} + \frac{b_2}{8 B_3} \bar{\alpha}_6 + 8 \bar{\alpha}_8 \\ &= -\frac{B_4 K}{4 B_7 \bar{B}_3} \left[24(3 B_1 - B_3) \alpha_6 \beta_3 + 16(3 B_1 - B_3) \alpha_4 \beta_5 + \right. \\ &\quad \left. + 8(2 B_1 - B_3) \alpha_2 \beta_7 - 4(10 B_1 - 3 B_3) \alpha_4^2 - 6(10 B_1 - \right. \\ &\quad \left. - 3 B_3) \alpha_2 \alpha_6 \right] + \frac{\bar{B}_1 B_4}{B_1 \bar{B}_3 B_7 K} \left[\left(3 B_1 + \frac{B_3}{2} \right) \left(\frac{1}{4} \alpha_2 \alpha_6 + \right. \right. \\ &\quad \left. \left. + \frac{1}{6} \alpha_4^2 \right) \right], \end{aligned} \quad (3-32m)$$

$$\begin{aligned} &9216 b_1 \bar{\alpha}_8 - 4 b_2 \bar{\alpha}_4 - 144(1 + \bar{B}_3) \bar{\alpha}_6 + \frac{\bar{B}_1 B_3}{4 B_1} \bar{\alpha}_2 \\ &= \frac{B_4 K}{B_7} \left\{ 144(2 B_1 - B_3) \beta_1 \alpha_6 + 16[2(8 B_1 - 3 B_3) \alpha_4 \beta_3 + \right. \\ &\quad \left. + 3(2 B_1 - B_3) \alpha_2 \beta_5 - 2(22 B_1 - 9 B_3) \alpha_2 \alpha_4] \right\} - \end{aligned}$$

$$- \left\{ 4(2B_1 - B_3)K\beta_1\alpha_4 + \frac{K}{2}(2B_1 - B_3)(4\alpha_2\beta_3 - 3\alpha_2^2) \right\} - \frac{4\bar{B}_1B_4}{B_1B_7K}(4B_1 + B_3)\alpha_2\alpha_4 + \frac{\bar{B}_1}{8KB_1}(2B_1 + B_3)\alpha_2^2, \quad (3-32n)$$

$$38400b_1\bar{\alpha}_{10} - 6b_2\bar{\alpha}_6 - 384(1 + \bar{B}_3)\bar{\alpha}_8 + \frac{\bar{B}_1B_3}{6B_1}\bar{\alpha}_4 \\ = \frac{24KB_4}{B_7} \left[(3B_1 - B_3)(12\alpha_6\beta_3 + 8\alpha_4\beta_5) + 4(2B_1 - B_3)\alpha_2\beta_7 - 6(10B_1 - 3B_3)\alpha_4^2 - 12(8B_1 - 3B_3)\alpha_2\alpha_6 \right] - 6(2B_1 - B_3)K\beta_1\alpha_6 - \frac{K}{3} \left[4(8B_1 - 3B_3)\alpha_4\beta_3 + 6(2B_1 - B_3)\alpha_2\beta_5 - 4(8B_1 - 3B_3)\alpha_2\alpha_4 \right] - \frac{\bar{B}_1B_4}{B_1B_7K} \left[4(6B_1 + B_3)\alpha_4^2 + 6(6B_1 + B_3)\alpha_2\alpha_6 \right] + \frac{\bar{B}_1}{6KB_1}(4B_1 + B_3)\alpha_2\alpha_4, \quad (3-32o)$$

$$\bar{\beta}_9 - \left(\frac{4}{5} + \frac{B_3}{10B_1} \right) \bar{\alpha}_8 = -\frac{3}{20KB_1}(8B_1 + B_3)\alpha_4\alpha_6, \quad (3-32p)$$

$$\bar{\beta}_{11} - \left(\frac{5}{6} + \frac{B_3}{12B_1} \right) \bar{\alpha}_{10} = -\frac{3}{40KB_1}(10B_1 + B_3)\alpha_6^2, \quad (3-32q)$$

$$\bar{\gamma}_9 + \frac{b_2}{10\bar{B}_3}\bar{\alpha}_8 + 10\bar{\alpha}_{10} = -\frac{B_4K}{5B_7\bar{B}_3} \left[6(18B_1 - 5B_3)\alpha_6\beta_5 + 4(16B_1 - 5B_3)\alpha_4\beta_7 - 36(4B_1 - B_3)\alpha_4\alpha_6 \right] + \frac{3\bar{B}_1B_4}{10B_1\bar{B}_3B_7K} \left(4B_1 + \frac{B_2}{2} \right) \alpha_4\alpha_6, \quad (3-32r)$$

$$\bar{Y}_{11} + \frac{\bar{I}_2}{12\bar{B}_3} \bar{\alpha}_{10} = -\frac{B_4 K}{2B_7 \bar{B}_3} \left[12(4B_1 - B_3) \alpha_6 \beta_7 - 3(4B_1 - 3B_3) \alpha_6^2 \right] + \frac{3\bar{B}_1 B_4}{10B_7 \bar{B}_3 B_7 K} \left(\frac{5}{2} B_1 + \frac{B_3}{4} \right) \alpha_6^2. \quad (3-32s)$$

These algebraic equations are solved by the second part of the same digital computer program in Fortran II (Appendix B). The results are listed in Tables 1 through 4 for various values of k which is the index of the shallowness of the structure. The dimensionless loading versus the dimensionless central deflection curves are shown in Figure 5.

B. Ordinary Shallow Spherical Shell (as a limiting case)

If we set the thickness of the core, $\bar{t}$, equal to zero the case is reduced to that of an ordinary shallow spherical shell. Thus $B_4 = B_5 = B_6 = B_7 = 0$, $B_3 = B_4 = 0$. The number of the differential equations of each set is reduced to two. For the first set of differential equations, we have

$$B_1 \left[\gamma \frac{d^2 U_1}{d\gamma^2} + \frac{dU_1}{d\gamma} - \frac{U_1}{\gamma} - \left(\gamma^2 \frac{d^2 W_1}{d\gamma^2} + \gamma \frac{dW_1}{d\gamma} \right) \right] - B_3 \gamma \frac{dW_1}{d\gamma} = 0, \quad (3-33a)$$

$$\bar{B}_1 \left[\eta \frac{dU}{d\eta} - \eta^2 \frac{dW_1}{d\eta} + U_1 \right] - \left(\eta \frac{d^4 W_1}{d\eta^4} + 2 \frac{d^3 W_1}{d\eta^3} - \frac{1}{\eta} \frac{d^2 W_1}{d\eta^2} + \frac{1}{\eta^2} \frac{dW_1}{d\eta} \right) + f_1 \eta = 0, \quad (3-33b)$$

where

$$\bar{B}_1 = (2B_3 - 4B_1)K^2,$$

and the boundary conditions

$$W_1 = 1, \quad U_1 = \frac{dW_1}{d\eta} = 0, \quad \text{at } \eta = 0 \quad (3-34a)$$

$$U_1 = W_1 = \frac{dW_1}{d\eta} = 0, \quad \text{at } \eta = 1. \quad (3-34b)$$

The second set of differential equations become

$$B_1 \left[\eta \frac{d^2 U_2}{d\eta^2} + \frac{dU_2}{d\eta} - \frac{U_2}{\eta} - \left(\eta^2 \frac{d^2 W_2}{d\eta^2} + \eta \frac{dW_2}{d\eta} \right) \right] + \frac{B_1}{2K} \eta \frac{dW_1}{d\eta} \frac{d^2 W_1}{d\eta^2} + \frac{B_3}{4K} \left(\frac{dW_1}{d\eta} \right)^2 - B_3 \eta \frac{dW_2}{d\eta} = 0, \quad (3-35a)$$

$$\begin{aligned} & \left\{ B_1 K \left(\eta \frac{dU_1}{d\eta} - \eta^2 \frac{dW_1}{d\eta} + U_1 \right) - B_3 K U_1 \right\} \frac{d^2 W_1}{d\eta^2} + \\ & + \left\{ (B_1 - B_3) K \left(\frac{dU_1}{d\eta} - \eta \frac{dW_1}{d\eta} \right) + B_1 K \frac{U_1}{\eta} \right\} \frac{dW_1}{d\eta} + \\ & + \bar{B}_1 \left[\frac{dU_2}{d\eta} + U_2 - \eta^2 \frac{dW_2}{d\eta} + \frac{1}{4K} \eta \left(\frac{dW_1}{d\eta} \right)^2 \right] - \left(\eta \frac{d^4 W_2}{d\eta^4} + 2 \frac{d^3 W_2}{d\eta^3} - \frac{1}{\eta} \frac{d^2 W_2}{d\eta^2} + \frac{1}{\eta^2} \frac{dW_2}{d\eta} \right) + f_2 \eta = 0, \end{aligned} \quad (3-35b)$$

and the boundary conditions are

$$U_2 = \frac{dW_2}{d\eta} = W_2 = 0, \quad \text{at } \eta = 0, \quad (3-36a)$$

$$U_2 = \frac{dW_2}{d\eta} = W_2 = 0, \quad \text{at } \eta = 1. \quad (3-36b)$$

From equation (3-33a), solving for U_1 , we have

$$\frac{d}{d\eta}(\eta U_1) = \eta^2 \frac{dW_1}{d\eta} + \frac{B_2}{B_1} \eta W_1 + C_1 \eta. \quad (3-37)$$

Substituting into Equation (3-33b), we obtain

$$\begin{aligned} \frac{\bar{B}_1 B_2}{B_1} W_1 + \bar{B}_1 C_1 \eta - \left(\eta \frac{d^4 W_1}{d\eta^4} + 2 \frac{d^3 W_1}{d\eta^3} - \frac{1}{\eta} \frac{d^2 W_1}{d\eta^2} \right. \\ \left. + \frac{1}{\eta^2} \frac{dW_1}{d\eta} \right) + f_1 \eta = 0. \end{aligned} \quad (3-38)$$

Let

$$W_1 = 1 + \sum_{p=1}^{\infty} \alpha_{2p} \eta^{2p}, \quad (3-39)$$

then

$$\begin{aligned} \frac{dW_1}{d\eta} &= \sum_{p=1}^{\infty} 2p \alpha_{2p} \eta^{2p-1}, \\ \frac{d^2 W_1}{d\eta^2} &= \sum_{p=1}^{\infty} 2p(2p-1) \alpha_{2p} \eta^{2p-2}, \\ \frac{d^3 W_1}{d\eta^3} &= \sum_{p=2}^{\infty} 2p(2p-1)(2p-2) \alpha_{2p} \eta^{2p-3}, \\ \frac{d^4 W_1}{d\eta^4} &= \sum_{p=2}^{\infty} 2p(2p-1)(2p-2)(2p-3) \alpha_{2p} \eta^{2p-4}. \end{aligned} \quad (3-40)$$

Substituting into Equation (3-38), we have the recurrence relation

$$\frac{\bar{B}_1 B_3}{B_1} + \bar{B}_1 c_1 - 64 \alpha_4 + q_1 = 0, \quad (3-41)$$

$$\frac{\bar{B}_1 B_3}{B_1} \alpha_{2p} - 16(p+2)^2(p+1)^2 \alpha_{2p+4} = 0, \quad (3-42)$$

for $p = 1, 2, 3, \dots$

Substituting Equation (3-39) into Equation (3-37)

$$U_1 = \sum_{p=1}^{\infty} \left[\frac{p}{p+1} + \frac{B_3}{2(p+1)B_1} \right] \alpha_{2p} \gamma^{2p+1} + \frac{B_3}{2B_1} \gamma + \frac{c_1}{2} \gamma + \frac{c_2}{\gamma}, \quad (3-43)$$

and $c_2 = 0$, by the boundary conditions (3-34a).

$$\text{Set } U_1 = \beta_1 \gamma + \sum_{p=1}^{\infty} \beta_{2p+1} \gamma^{2p+1}, \quad (3-44)$$

where

$$\beta_1 = \frac{B_3}{2B_1} + \frac{c_1}{2} \quad (3-45)$$

$$\beta_{2p+1} = \left[\frac{p}{p+1} + \frac{B_3}{2(p+1)B_1} \right] \alpha_{2p}, \quad (3-46)$$

for $p = 1, 2, 3, \dots$

With the help of the boundary conditions (3-34b), we can determine the unknowns, α_2 , c_1 and q_1 . Consequently W_1 and U_1 are obtained.

The same procedure is applied to solve for W_2 and U_2 .

Let

$$\begin{aligned}\chi(\eta) &= \frac{B_1}{2K} \eta \frac{dW_1}{d\eta} \frac{d^2 W_1}{d\eta^2} + \frac{B_3}{4K} \left(\frac{dW_1}{d\eta} \right)^2 \\ &= \sum_{\beta=1}^{\infty} \sum_{s=1}^{\infty} \left[\frac{2B_1}{K} (2s-1) + \frac{B_3}{K} \right] \beta^s \alpha_{2\beta} \alpha_{2s} \eta^{2(\beta+s-1)}, \quad (3-47)\end{aligned}$$

$$\begin{aligned}\gamma(\eta) &= \left\{ B_1 K \eta \frac{dU_1}{d\eta} - B_1 K \eta^2 \frac{dW_1}{d\eta} + (B_1 - B_3) K U_1 \right\} \frac{dW_1}{d\eta^2} + \\ &+ \left\{ (B_1 - B_3) K \left(\frac{dU_1}{d\eta} - \eta \frac{dW_1}{d\eta} \right) + B_1 K \frac{U_1}{\eta} \right\} \frac{dW_1}{d\eta} + \\ &+ \frac{\bar{B}_1}{4K} \eta \left(\frac{dW_1}{d\eta} \right)^2 \\ &= \sum_{\beta=1}^{\infty} 4\beta^2 (2B_1 - B_3) K \beta_1 \alpha_{2\beta} \eta^{2\beta-1} + \sum_{\beta=2}^{\infty} \sum_{s=1}^{\infty} \left\{ \right. \\ &4SK [2s\beta B_1 - (s+\beta-1)B_3] \alpha_{2s} \beta_{2\beta-1} - 2SKX \\ &\left. (\beta-1) [2(2s+1)B_1 - 3B_3] \alpha_{2s} \alpha_{2\beta-2} \right\} \eta^{2(\beta+s)-3}. \quad (3-48)\end{aligned}$$

From Equation (3-35a) we solve for U_2 as

$$\begin{aligned}U_2 &= \frac{1}{\eta} \int \eta^2 \frac{dW_2}{d\eta} d\eta + \frac{B_3}{B_1} \frac{1}{\eta} \int \eta W_2 d\eta - \\ &- \frac{1}{B_1 \eta} \int \eta \int \frac{1}{\eta} \chi(\eta) d\eta d\eta + \frac{\bar{C}_1}{2} \eta \\ &+ \frac{\bar{C}_3}{\eta}.\end{aligned} \quad (3-49)$$

Let W_2 be also expanded into power-series

$$W_2 = \sum_{p=1}^{\infty} \bar{\alpha}_{2p} \gamma^{2p} \quad (3-50)$$

By substituting of Equations (3-49) and (3-50) into Equation (3-35b) and equating the coefficients of the same order terms, we get the following formulae

$$\frac{\bar{g}_2}{2} - 32\bar{\alpha}_4 + \frac{\bar{B}_1}{2}\bar{c}_1 = (2B_3 - 4B_1)K\beta_1\alpha_2, \quad (3-51a)$$

$$\begin{aligned} & \frac{\bar{B}_1 B_3}{2mB_1} \bar{\alpha}_{2m-2} - 2m(2m+2)^2 \bar{\alpha}_{2m+2} \\ &= - \left\{ 2m(2B_1 - B_3)K\beta_1\alpha_{2m} + \sum_{p=2}^m \frac{K}{m} \left\{ 2(m-p+1) \right. \right. \\ & \quad \left. \left. \left[2p(m-p+1)B_1 - mB_3 \right] \alpha_{2(m-p+1)} \beta_{2p-1} - \right. \right. \\ & \quad \left. \left. - (m-p+1)(p-1) \left[2(2m-2p+3)B_1 - 3B_3 \right] \alpha_{2(m-p+1)} \right. \right. \\ & \quad \left. \left. \alpha_{2p-2} \right\} \right\} + \frac{\bar{B}_1}{4KB_1} \sum_{p=1}^{m-1} \frac{p(m-p)}{m(m-1)} \left[2(2m-2p-1) \right. \\ & \quad \left. B_1 + B_3 \right] \alpha_{2p} \alpha_{2(m-p)}, \\ & \text{for } m = 2, 3, 4, \dots \end{aligned} \quad (3-51b)$$

Substituting Equation (3-50) into Equation (3-49) and applying the boundary condition (3-36a), we have $\bar{c}_3 = 0$ and

$$U_2 = \sum_{p=0}^{\infty} \bar{\beta}_{2p+1} \gamma^{2p+1}, \quad (3-52)$$

where

$$\bar{\beta}_1 = \frac{\bar{C}_1}{2}, \quad (3-53a)$$

$$\begin{aligned} \bar{\beta}_{2p+1} = & \left[\frac{p}{\beta+1} + \frac{B_2}{2(\beta+1)B_1} \right] \bar{\alpha}_{2p} - \frac{1}{B_1} \sum_{m=1}^p \left[\right. \\ & \left. \frac{B_1}{2K} (2p-2m+1) + \frac{B_2}{4K} \right] \alpha_{2m} \alpha_{2(p-m+1)}. \end{aligned} \quad (3-53b)$$

By applying the boundary conditions (3-36b), these three unknowns, $\bar{\alpha}_2$, $\bar{c}_1$ and q_2 can be determined.

Let us for a rough approximation assume

$$W_1 = 1 + \alpha_2 \eta^2 + \alpha_4 \eta^4, \quad (3-54a)$$

$$U_1 = \beta_1 \eta + \beta_3 \eta^3 + \beta_5 \eta^5, \quad (3-54b)$$

$$W_2 = \bar{\alpha}_2 \eta^2 + \bar{\alpha}_4 \eta^4 + \bar{\alpha}_6 \eta^6 + \bar{\alpha}_8 \eta^8 + \bar{\alpha}_{10} \eta^{10}, \quad (3-55a)$$

$$U_2 = \bar{\beta}_1 \eta + \bar{\beta}_3 \eta^3 + \bar{\beta}_5 \eta^5 + \bar{\beta}_7 \eta^7 + \bar{\beta}_9 \eta^9 + \bar{\beta}_{11} \eta^{11}. \quad (3-55b)$$

By the formulae (3-34b), (3-42), (3-41), (3-45), (3-46) and (3-54) we obtain the first set of algebraic equations

$$\frac{q_1}{2} - 32\alpha_4 + \frac{B_1}{2} C_1 = -\frac{\bar{B}_1 B_2}{2 B_1}, \quad (3-56a)$$

$$\alpha_2 + \alpha_4 = -1, \quad (3-56b)$$

$$\alpha_2 + 2\alpha_4 = 0, \quad (3-56c)$$

$$\beta_1 + \beta_3 + \beta_5 = 0, \quad (3-56d)$$

$$\beta_3 - \left(\frac{1}{2} + \frac{B_2}{4B_1} \right) \alpha_2 = 0, \quad (3-56e)$$

$$\beta_5 - \left(\frac{2}{3} + \frac{B_3}{6B_1}\right)\alpha_4 = 0, \quad (3-56f)$$

$$\beta_1 - \frac{C_L}{2} = \frac{B_3}{2B_1} \quad (3-56g)$$

By formulae (3-36b), (3-51), (3-53) and (3-55) we obtain the second set of algebraic equations

$$\frac{I_2}{2} - 32\bar{\alpha}_4 + \frac{\bar{B}_1}{2}\bar{C}_1 = 2K(B_3 - 2B_1)\beta_1\alpha_2, \quad (3-57a)$$

$$\frac{\bar{B}_1 B_1}{4B_1}\bar{\alpha}_2 - 144\bar{\alpha}_6 = \left[\frac{3}{2}K(2B_1 - B_3) + \frac{\bar{B}_1}{8B_1 K}(2B_1 + B_3)\right]\alpha_2^2 - K[4(2B_1 - B_3)\beta_1\alpha_4 + 2(2B_1 - B_3)\alpha_2\beta_3], \quad (3-57b)$$

$$\bar{\alpha}_2 + \bar{\alpha}_4 + \bar{\alpha}_6 + \bar{\alpha}_8 + \bar{\alpha}_{10} = 0, \quad (3-57c)$$

$$\bar{\alpha}_2 + 2\bar{\alpha}_4 + 3\bar{\alpha}_6 + 4\bar{\alpha}_8 + 5\bar{\alpha}_{10} = 0, \quad (3-57d)$$

$$\bar{\beta}_1 + \bar{\beta}_3 + \bar{\beta}_5 + \bar{\beta}_7 + \bar{\beta}_9 + \bar{\beta}_{11} = 0, \quad (3-57e)$$

$$\bar{\beta}_3 - \left(\frac{1}{2} + \frac{B_3}{4B_1}\right)\bar{\alpha}_2 = -\frac{1}{2B_1}\left(\frac{B_1}{2K} + \frac{B_3}{4K}\right)\alpha_2^2, \quad (3-57f)$$

$$\bar{\beta}_5 - \left(\frac{2}{3} + \frac{B_3}{6B_1}\right)\bar{\alpha}_4 = \frac{-1}{3B_1}\left(\frac{2B_1}{K} + \frac{B_3}{2K}\right)\alpha_2\alpha_4, \quad (3-57g)$$

$$\bar{\beta}_7 - \left(\frac{3}{4} + \frac{B_3}{8B_1}\right)\bar{\alpha}_6 = -\frac{1}{B_1}\left[\frac{1}{4}\left(\frac{3B_1}{K} + \frac{B_3}{2K}\right)\alpha_2\alpha_6 + \frac{1}{3}\left(\frac{3B_1}{2K} + \frac{B_3}{4K}\right)\alpha_4^2\right], \quad (3-57h)$$

$$\bar{\beta}_1 - \frac{\bar{\epsilon}_L}{2} = 0, \quad (3-571)$$

$$\begin{aligned} \frac{\bar{B}_1 B_2}{6 B_1} \bar{\alpha}_4 - 384 \bar{\alpha}_8 = & -\frac{K}{3} \left[4(8B_1 - 3B_3) \alpha_4 \beta_3 + \right. \\ & \left. + 6(2B_1 - B_3) \alpha_2 \beta_5 - 4(8B_1 - 3B_3) \alpha_2 \alpha_4 \right] + \\ & + \frac{\bar{B}_1}{6 B_1 K} (4B_1 + B_3) \alpha_2 \alpha_4, \end{aligned} \quad (3-57j)$$

$$\bar{\beta}_9 - \left(\frac{4}{5} + \frac{B_3}{10 B_1} \right) \bar{\alpha}_8 = 0, \quad (3-57k)$$

$$\begin{aligned} \frac{\bar{B}_1 B_3}{8 B_1} \bar{\alpha}_6 - 800 \bar{\alpha}_{10} = & -K \left[4(3B_1 - B_3) \alpha_4 \beta_5 - \right. \\ & \left. - \frac{3}{2}(10B_1 - 3B_3) \alpha_4^2 \right] + \frac{\bar{B}_1}{12 B_1 K} (6B_1 + B_3) \alpha_4^2, \end{aligned} \quad (3-57l)$$

$$\bar{\beta}_{11} - \left(\frac{5}{6} + \frac{B_3}{12 B_1} \right) \bar{\alpha}_{10} = 0. \quad (3-57m)$$

A computer program in Fortran II (Appendix C) is used to solve Equation (3-56) and (3-57), the results are plotted as dimensionless loading versus dimensionless central deflection and compared with known solutions (23, 24), as shown in Figure 6.

CHAPTER IV

CONCLUSIONS

Both the derivation and a numerical application for a large-deflection theory of orthotropic sandwich shallow shells have been carried out in this dissertation. The numerical solution to a typical engineering problem is not only to demonstrate the applicability of the theory proposed here but also to show that the numerical procedure adopted is useful in solving such sandwich shell problems.

In the theoretical derivation the procedure is general enough to cover all the cases of orthotropic sandwich shallow shell structures. All the quantities involved are expressed in general coordinates so that one can easily apply them to any specific geometrical configuration by the proper coordinate transformation. The derivation is based on the following considerations.

(1) The structure is shallow; i.e., the rise of its middle surface, z , and its first as well as second derivatives are so small that their products are negligible in comparison to other dimensions.

2) The facings of the structure are thin enough that Kirchhoff-Love hypothesis is valid.

3) The transverse shear deformation affect is included in the core.

4) The planes normal to the middle surface remain plane but no longer normal to the middle surface after deformation.

5) No transverse compression through the thickness is considered, i.e., the deflection w is independent of the transverse coordinate.

6) A generalized plane-stress theory is applied.

7) The temperature distribution is a known function of the space coordinates.

Most of these considerations are commonly used in ordinary shell theory except for items (3) and (4) which distinguish sandwich shell theory from classical shell theory.

For the numerical solution, the temperature terms are dropped for simplicity. However, this does not cause any difference in the numerical method to be used since the temperature is a known function which effects only a finite number of terms in the series solution and the recurrence formulae still hold.

No effort has been made to improve the accuracy of the numerical result, since this can be done easily by increasing the number of terms in the expansion. Even

for a rough approximation, the result is within 10 per cent of known solutions. The problem is completely solved once the deflection w and the displacement u and the angular rotation θ are determined.

TABLE 1

COEFFICIENTS FOR SANDWICH SHALLOW SPHERICAL SHELL

$$k = 2.0$$

α_0	1.0	β_1	0.53	γ_1	5.23
α_2	-2.70	β_3	-1.82	γ_3	-10.3
α_4	2.55	β_5	1.99	γ_5	5.19
α_6	-0.85	β_7	-0.71	γ_7	-0.03

$\bar{\alpha}_0$	0.0	$\bar{\beta}_1$	0.24	$\bar{\gamma}_1$	-2.15
$\bar{\alpha}_2$	1.16	$\bar{\beta}_3$	-0.44	$\bar{\gamma}_3$	10.39
$\bar{\alpha}_4$	-2.81	$\bar{\beta}_5$	0.49	$\bar{\gamma}_5$	-11.37
$\bar{\alpha}_6$	2.05	$\bar{\beta}_7$	-1.05	$\bar{\gamma}_7$	0.50
$\bar{\alpha}_8$	-0.17	$\bar{\beta}_9$	1.25	$\bar{\gamma}_9$	2.68
$\bar{\alpha}_{10}$	-0.24	$\bar{\beta}_{11}$	-0.50	$\bar{\gamma}_{11}$	-0.05

TABLE 2

COEFFICIENTS FOR SANDWICH SHALLOW SPHERICAL SHELL

$$k = 1.5$$

α_0	1.0	β_1	0.53	γ_1	5.31
α_2	-2.74	β_3	-1.84	γ_3	-10.56
α_4	2.61	β_5	2.04	γ_5	5.27
α_6	-0.87	β_7	-0.72	γ_7	-0.02

$\bar{\alpha}_0$	0.0	$\bar{\beta}_1$	0.29	$\bar{\gamma}_1$	-1.63
$\bar{\alpha}_2$	0.88	$\bar{\beta}_3$	-1.09	$\bar{\gamma}_3$	8.08
$\bar{\alpha}_4$	-2.19	$\bar{\beta}_5$	2.02	$\bar{\gamma}_5$	-9.18
$\bar{\alpha}_6$	1.66	$\bar{\beta}_7$	-2.47	$\bar{\gamma}_7$	0.78
$\bar{\alpha}_8$	-0.18	$\bar{\beta}_9$	1.81	$\bar{\gamma}_9$	1.98
$\bar{\alpha}_{10}$	-0.17	$\bar{\beta}_{11}$	-0.56	$\bar{\gamma}_{11}$	-0.04

TABLE 3

COEFFICIENTS FOR SANDWICH SHALLOW SPHERICAL SHELL

$$k = 1.0$$

α_0	1.0	β_1	0.53	γ_1	5.37
α_2	-2.77	β_3	-1.86	γ_3	-10.67
α_4	2.65	β_5	2.07	γ_5	5.32
α_6	-0.88	β_7	-0.74	γ_7	-0.01
$\bar{\alpha}_0$	0.0	$\bar{\beta}_1$	0.40	$\bar{\gamma}_1$	-1.04
$\bar{\alpha}_2$	0.57	$\bar{\beta}_3$	-2.20	$\bar{\gamma}_3$	5.42
$\bar{\alpha}_4$	-1.47	$\bar{\beta}_5$	4.59	$\bar{\gamma}_5$	-6.44
$\bar{\alpha}_6$	1.17	$\bar{\beta}_7$	4.59	$\bar{\gamma}_7$	0.78
$\bar{\alpha}_8$	-0.15	$\bar{\beta}_9$	2.91	$\bar{\gamma}_9$	1.30
$\bar{\alpha}_{10}$	-0.12	$\bar{\beta}_{11}$	-0.72	$\bar{\gamma}_{11}$	-0.03

TABLE 4

COEFFICIENTS FOR SANDWICH SHALLOW SPHERICAL SHELL

$$k = 0.5$$

α_0	1.0	β_1	0.52	γ_1	5.40
α_2	-2.79	β_3	-1.87	γ_3	-10.75
α_4	2.68	β_5	2.09	γ_5	5.35
α_6	-0.89	β_7	-0.74	γ_7	-0.002
$\bar{\alpha}_0$	0.0	$\bar{\beta}_1$	0.76	$\bar{\gamma}_1$	-0.41
$\bar{\alpha}_2$	0.22	$\bar{\beta}_3$	-5.08	$\bar{\gamma}_3$	2.51
$\bar{\alpha}_4$	-0.69	$\bar{\beta}_5$	11.16	$\bar{\gamma}_5$	-3.32
$\bar{\alpha}_6$	0.60	$\bar{\beta}_7$	-11.65	$\bar{\gamma}_7$	0.57
$\bar{\alpha}_8$	-0.08	$\bar{\beta}_9$	6.14	$\bar{\gamma}_9$	0.66
$\bar{\alpha}_{10}$	-0.06	$\bar{\beta}_{11}$	-1.32	$\bar{\gamma}_{11}$	-0.01

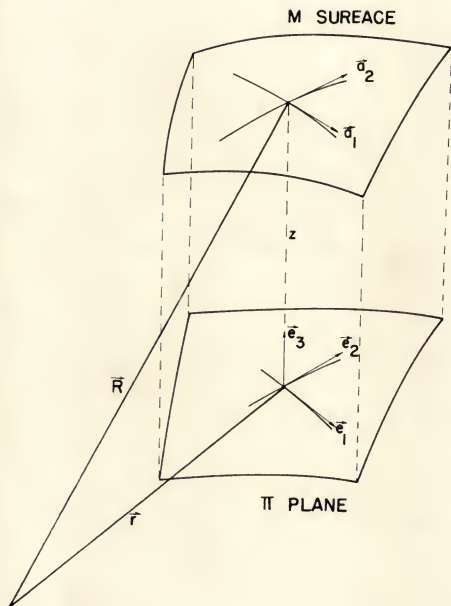


FIG.1 PROJECTION OF M-SURFACE ON π -PLANE

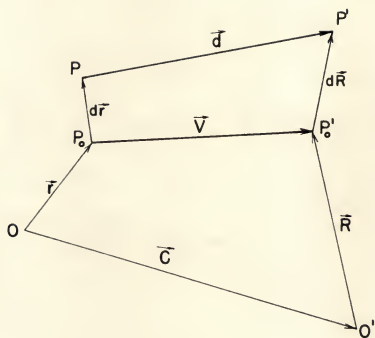


FIG. 2 DISPLACEMENT VECTORS

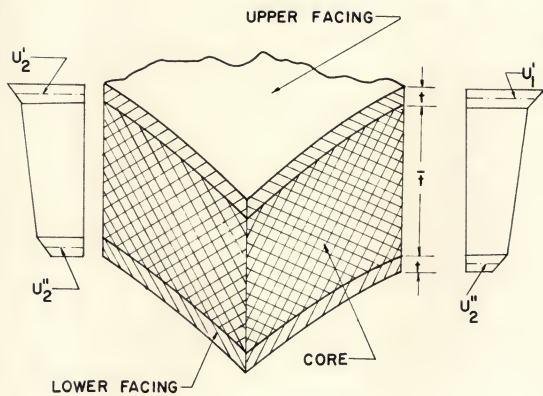


FIG. 3 SANDWICH STRUCTURE

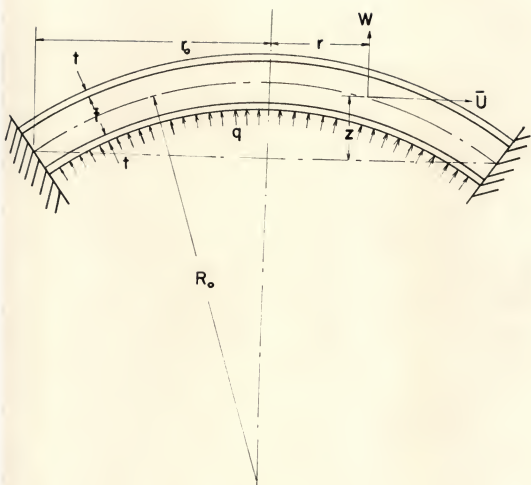


FIG. 4 SANDWICH SHALLOW SPHERICAL SHELL

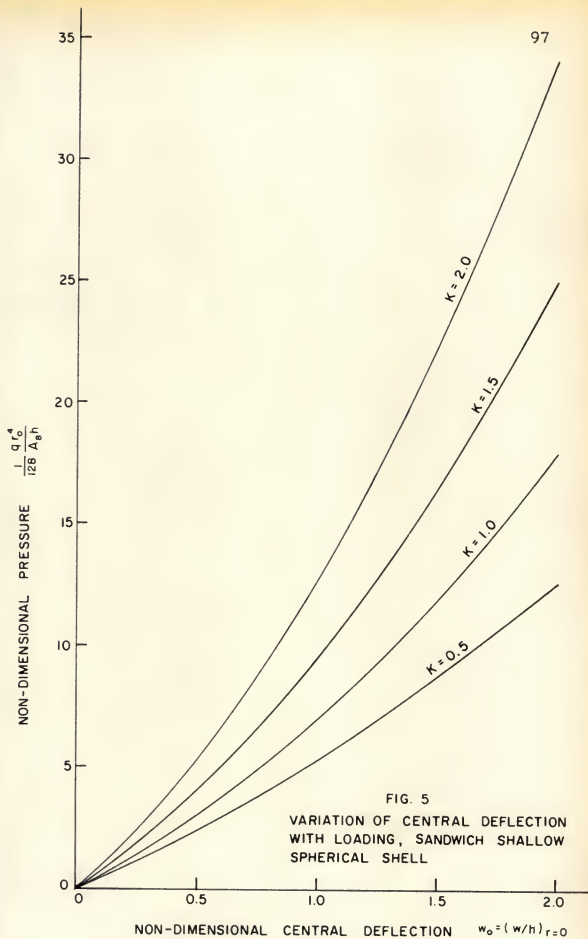
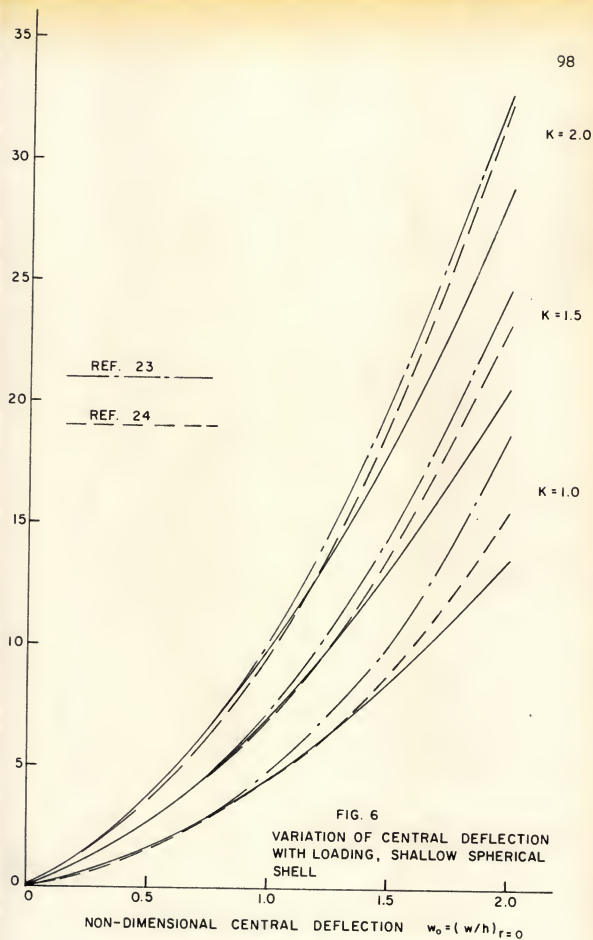


FIG. 5
VARIATION OF CENTRAL DEFLECTION
WITH LOADING, SANDWICH SHALLOW
SPHERICAL SHELL



APPENDICES

APPENDIX A

APPENDIX A

PROOF OF THE CONVERGENCE OF THE POWER SERIES

From the recurrence formula (3-18), solving for α_{2p+4}

$$\alpha_{2p+4} = \frac{1}{2(p+2)^2 b_1} \alpha_{2p+2} + \frac{b_4}{2^4(p+2)^2(p+1)^2 b_1} \alpha_{2p} - \frac{b_3}{2^6(p+2)^2(p+1)^2 p^2 b_1} \alpha_{2p-2}, \quad (\text{A-1})$$

where b_1 , b_2 and b_3 are defined by Equation (3-20n). Since η is in the interval of $[0, 1]$, let us only consider the ratio test of α 's.

Replacing p by $p + 1$ and by substitution of Equation (A-1), we have the coefficient for the next term

$$\alpha_{2p+6} = \frac{(\frac{1}{b_1^2} + \frac{b_2}{4b_1})}{2^2(p+3)^2(p+2)^2} \alpha_{2p+2} + \left[\frac{b_2}{2^5(p+3)^2(p+2)^2(p+1)^2 b_1} - \frac{b_3}{2^6(p+3)^2(p+2)^2(p+1)^2 b_1} \right] \alpha_{2p} - \frac{b_3}{2^7(p+3)^2(p+2)^2 p^2 b_1^2} \alpha_{2p-2}. \quad (\text{A-2})$$

The ratio is

$$\frac{\alpha_{2p+6}}{\alpha_{2p+4}} = \left[\frac{1}{2(p+3)^2} \right] \cdot \left\{ \left(\frac{1}{b_1} + \frac{b_2}{4} \right) \alpha_{2p+2} + \right. \\ \left. + \left(\frac{b_2}{b_1} - \frac{b_3}{2} \right) \frac{\alpha_{2p}}{8(p+1)^2} - \frac{b_3}{32(p+1)^2 p^2 b_1} \alpha_{2p-2} \right\} / \\ \left\{ \alpha_{2p+2} + \frac{b_2}{8(p+1)^2} \alpha_{2p} - \frac{b_3}{32(p+1)^2 p^2} \alpha_{2p-2} \right\}. \quad (A-3)$$

There are two possibilities:

Case 1

$$|\alpha_{2p-2}| > |\alpha_{2p}| > |\alpha_{2p+2}|$$

Dividing both the numerator and denominator by α_{2p-2} , we have the ratio

$$\frac{\alpha_{2p+6}}{\alpha_{2p+4}} = \left[\frac{1}{2(p+3)^2} \right] \cdot \left\{ \left(\frac{1}{b_1} + \frac{b_2}{4} \right) \frac{\alpha_{2p+2}}{\alpha_{2p-2}} + \right. \\ \left. \frac{\left(\frac{b_2}{b_1} - \frac{b_3}{2} \right)}{8(p+1)^2} \frac{\alpha_{2p}}{\alpha_{2p-2}} - \frac{b_3}{32(p+1)^2 p^2 b_1} \right\} / \\ \left\{ \frac{\alpha_{2p+2}}{\alpha_{2p-2}} + \frac{b_2}{8(p+1)^2} \frac{\alpha_{2p}}{\alpha_{2p-2}} - \frac{b_3}{32(p+1)^2 p^2} \right\}. \quad (A-4)$$

Let us take the limit of Equation (A-4) as p becomes very large. Suppose $|\alpha_{2p+2}|/|\alpha_{2p-2}|$ does not approach to zero as $p \rightarrow \infty$. Otherwise the series is convergent by this condition. Since $\left| \frac{\alpha_{2p+2}}{\alpha_{2p-2}} \right| = \left| \frac{\alpha_{2p+2}}{\alpha_{2p}} \right| \left| \frac{\alpha_{2p}}{\alpha_{2p-2}} \right| \rightarrow 0$ as $p \rightarrow \infty$ implies $\left| \frac{\alpha_{2p+2}}{\alpha_{2p}} \right| \rightarrow 0 < 1$, as $p \rightarrow \infty$.

$$\begin{aligned}
\lim_{p \rightarrow \infty} \left| \frac{\alpha_{2p+6}}{\alpha_{2p+4}} \right| &= \lim_{p \rightarrow \infty} \left| \frac{1}{2(p+3)^2} \right| \cdot \lim_{p \rightarrow \infty} \left| \left\{ \left(\frac{1}{b_1} + \frac{b_2}{4} \right) \frac{\alpha_{2p+2}}{\alpha_{2p-2}} + \right. \right. \\
&\quad \left. \left. + \frac{\left(\frac{b_2}{b_1} - \frac{b_3}{2} \right)}{8(p+1)^2} \frac{\alpha_{2p}}{\alpha_{2p-2}} - \frac{b_3}{32(p+1)^2 p^2 b_1} \right\} \right| \\
&= \lim_{p \rightarrow \infty} \left| \frac{\alpha_{2p+2}}{\alpha_{2p-2}} + \frac{b_2}{8(p+1)^2} \frac{\alpha_{2p}}{\alpha_{2p-2}} - \frac{b_3}{32(p+1)^2 p^2} \right| \\
&= \lim_{p \rightarrow \infty} \left| \frac{1}{2(p+3)^2} \right| \cdot \left| \frac{1}{b_1} + \frac{b_2}{4} \right|. \quad (A-5)
\end{aligned}$$

Since b_1 , b_2 and b_3 are fixed constants which are independent of p , at a large value of p we can have

$$\lim_{p \rightarrow \infty} \left| \frac{\alpha_{2p+6}}{\alpha_{2p+4}} \right| = 0 < 1. \quad (A-6)$$

Hence, in this case, the series is convergent by the ratio test.

Case 2 $\alpha_{2p+2} > \alpha_{2p} > \alpha_{2p-2}$

Let us divide the numerator and denominator by

$$\alpha_{2p+2}$$

$$\begin{aligned}
\frac{\alpha_{2p+6}}{\alpha_{2p+4}} &= \frac{1}{2(p+3)^2} \cdot \left\{ \left(\frac{1}{b_1} + \frac{b_2}{4} \right) + \frac{\left(\frac{b_2}{b_1} - \frac{b_3}{2} \right)}{8(p+1)^2} \frac{\alpha_{2p}}{\alpha_{2p+2}} - \right. \\
&\quad \left. - \frac{b_3}{32(p+1)^2 p^2 b_1} \frac{\alpha_{2p-2}}{\alpha_{2p+2}} \right\} \bigg/ \left\{ 1 + \frac{b_2}{8(p+1)^2} \frac{\alpha_{2p}}{\alpha_{2p+2}} - \right. \\
&\quad \left. - \frac{b_3}{32(p+1)^2 p^2} \frac{\alpha_{2p-2}}{\alpha_{2p+2}} \right\}. \quad (A-7)
\end{aligned}$$

Taking the limit of its absolute value as p becomes very large, we have

$$\begin{aligned}
 \lim_{p \rightarrow \infty} \left| \frac{\alpha_{2p+6}}{\alpha_{2p+4}} \right| &= \lim_{p \rightarrow \infty} \left| \frac{1}{2(p+3)^2} \right| \lim_{p \rightarrow \infty} \left| \left\{ \left(\frac{1}{b_1} + \frac{b_2}{4} \right) + \right. \right. \\
 &\quad \left. \left. + \frac{\left(\frac{b_2}{b_1} - \frac{b_3}{2} \right)}{8(p+1)^2} \frac{\alpha_{2p}}{\alpha_{2p+2}} - \frac{b_3}{32(p+1)^2 p^2 b_1} \frac{\alpha_{2p-2}}{\alpha_{2p+2}} \right\} \right| \\
 &\quad \left| \left\{ 1 + \frac{b_2}{8(p+1)^2} \frac{\alpha_{2p}}{\alpha_{2p+2}} - \frac{b_3}{32(p+1)^2 p^2} \frac{\alpha_{2p-2}}{\alpha_{2p+2}} \right\} \right| \\
 &= \lim_{p \rightarrow \infty} \left| \frac{1}{2(p+3)^2} \right| \cdot \left| \frac{1}{b_1} + \frac{b_2}{4} \right|. \quad (A-8)
 \end{aligned}$$

By the same argument, we can see that the series is convergent also for Case 2.

APPENDIX B

COMPUTER PROGRAM 2
SANDWICH SHALLOW SPHERICAL SHELL

DIMENSION A(25,50), B(25,50), C(25,25), F(25), G(15),
H(25),X(10)

1 READ INPUT TAPE 5,2,M,(X(I), I=1,M)

2 FORMAT (1I5, 9F8.3)

WRITE OUTPUT TAPE 6,3,M, (X(I), I=1,M)

3 FORMAT (///40X,5H DATA//40X, I4/(36X,E14.4))

$BA = (2.*X(3) - 4.*X(1)) * X(8) * X(8)$

$BB = 2.*(X(1) + X(2) - X(3)) * X(8) * X(8)$

$BC = X(4) + X(6)$

$BD = X(5) + X(6)$

$BE = X(4) / X(7) - X(6) * BC / X(7)$

$BF = BA * X(3) * X(4) / (X(1) * X(7))$

$BQ = (0.5 * X(1) + .25 * X(3)) / X(8)$

$YA = 2.*X(1) - X(3)$

$YB = 4.*X(1) - X(3)$

$YC = YB + 2.*X(1)$

$YD = YC + 2.*X(1)$

$YE = 10.*X(1) - 3.*X(3)$

$YF = BA * X(4) / (BC * X(7))$

$YG = X(6) - 1.$

$YH = BA * X(4) / (X(1) * X(7) * X(8))$

C SET UP FIRST MATRIX

N=13

```

DO 10 I=1,N
DO 10 J=1,N
10 A(I,J)=0
A(1,1)=0.5
A(1,2)=-2.*BF
A(1,3)=-32.*(1.+BC)
A(1,4)=1152.*BE
A(1,9)=.5*BA
A(2,2)=1.
A(2,3)=1.
A(2,4)=1.
A(3,2)=.25*BF
A(3,3)=4.*BC
A(3,4)=-144.*BE
A(3,11)=BC
A(4,2)=2.*YG*(1.+X(9))
A(4,3)=4.*YG*(3.+X(9))
A(4,4)=6.*YG*(5.+X(9))
A(4,10)=X(6)
A(4,11)=3.*X(6)
A(4,12)=5.*X(6)
A(4,13)=7.*X(6)
A(5,5)=1.
A(5,6)=1.
A(5,7)=1.
A(5,8)=1.
A(6,2)=- (0.5+0.25*X(3)/X(1))

```

$A(6,6)=1.$

$A(7,3)=-(.2./3. +X(3)/(6.*X(1)))$

$A(7,7)=1.$

$A(8,4)=-(.75+.125*X(3)/X(1))$

$A(8,8)=1.$

$A(9,5)=1.$

$A(9,9)=-0.5$

$A(10,1)=.5*YF/BA$

$A(10,2)=2.$

$A(10,3)=-32.*BE/BC$

$A(10,9)=.5*YF$

$A(10,10)=1.$

$A(11,10)=1.$

$A(11,11)=1.$

$A(11,12)=1.$

$A(11,13)=1.$

$A(12,3)=BF/(6.*BC)$

$A(12,4)=6.$

$A(12,12)=1.$

$A(13,4)=.125*BF/BC$

$A(13,13)=1.$

CALL INVERT (N,A,C)

DO 15 I=1,N

15 F(I)=0

$F(1)=-.5*BA*X(3)/X(1)$

$F(2)=-1.$

$F(9)=.5*X(3)/X(1)$

F(10)=-.5*BF/BC

DO 20 I=1,N

SUMA=0.

DO 19 J=1,N

19 SUMA=SUMA+C(I,J)*F(J)

20 G(I)=SUMA

WRITE OUTPUT TAPE 6,4,N,(G(I), I=1,N)

4 FORMAT (///45X,5HFIRSTIX,3HSETIX,12HCOEFFICIENTS//10X,
15/(20X,4E20

1.8))

C SET UP SECOND MATRIX

N=19

DO 35 I=1,N

DO 35 J=1,N

35 A(I,J)=0

A(1,1)=0.5

A(1,2)=-2.*BF

A(1,3)=-32.*(1.+BC)

A(1,4)=1152.*BE

A(1,9)=.5*BA

A(2,2)=1.

A(2,3)=1.

A(2,4)=1.

A(2,14)=1.

A(2,15)=1.

A(3,2)=.25*BF

A(3,3)=4.*BC

A(3,4)=-144.*BE

$$A(3,11)=BC$$

$$A(4,2)=2.*YG*(1.+X(9))$$

$$A(4,3)=4.*YG*(3.+X(9))$$

$$A(4,4)=6.*YG*(5.+X(9))$$

$$A(4,10)=X(6)$$

$$A(4,11)=3.*X(6)$$

$$A(4,12)=5.*X(6)$$

$$A(4,13)=7.*X(6)$$

$$A(4,14)=8.*YG*(7.+X(9))$$

$$A(4,15)=10.*YG*(9.+X(9))$$

$$A(4,18)=9.*X(6)$$

$$A(4,19)=11.*X(6)$$

$$A(5,5)=1.$$

$$A(5,6)=1.$$

$$A(5,7)=1.$$

$$A(5,8)=1.$$

$$A(5,16)=1.$$

$$A(5,17)=1.$$

$$A(6,2)=- (0.5+0.25*X(3)/X(1))$$

$$A(6,6)=1.$$

$$A(7,3)=- (2./3. +X(3)/(6.*X(1)))$$

$$A(7,7)=1.$$

$$A(8,4)=- (.75+.125*X(3)/X(1))$$

$$A(8,8)=1.$$

$$A(9,5)=1.$$

$$A(9,9)=-0.5$$

$$A(10,1)=.5*YF/BA$$

$$A(10,2)=2.$$

$$A(10,3)=-32.*BE/BC$$

$$A(10,9)=-.5*YF$$

$$A(10,10)=1.$$

$$A(11,10)=1.$$

$$A(11,11)=1.$$

$$A(11,12)=1.$$

$$A(11,13)=1.$$

$$A(11,18)=1.$$

$$A(11,19)=1.$$

$$A(12,3)=BF/(6.*BC)$$

$$A(12,4)=6.$$

$$A(12,12)=1.$$

$$A(12,14)=-384.*BE/BC$$

$$A(13,4)=-.125*BF/BC$$

$$A(13,13)=1.$$

$$A(13,14)=8.$$

$$A(13,15)=-800.*BE/BC$$

$$A(14,2)=-.25*BA*X(3)/X(1)$$

$$A(14,3)=-4.*BF$$

$$A(14,4)=-144.*(1.+BC)$$

$$A(14,14)=9216.*BE$$

$$A(15,3)=BA*X(3)/(6.*X(1))$$

$$A(15,4)=-6.*BF$$

$$A(15,14)=-384.*(1.+BC)$$

$$A(15,15)=38400.*BE$$

$$A(16,14)=-(.8+.1*X(3)/X(1))$$

A(16,16)=1.

A(17,15)=-(5./6.+X(3)/(12.*X(1)))

A(17,17)=1.

A(18,14)=.1*BF/BC

A(18,15)=10.

A(18,18)=1.

A(19,15)=BF/(12.*BC)

A(19,19)=1.

CALL INVERT (N,A,C)

DO 40 I=1,N

40 F(1)=0

F(1)=4.*X(8)*X(4)*YA*(8.*G(5)*G(3)+4.*G(2)*G(6)-3.*G(2)*G(2))/

1X(7)-2.*X(8)*YA*G(5)*G(2)-2.*YH*(X(1)+.5*X(4))*G(2)*G(2)

F(3)=-X(4)*X(8)*YA*(4.*G(5)*G(3)+2.*G(2)*G(6)-1.5*G(2)*G(2))/X(7)

1+.5*YH*(.5*X(1)+.25*X(3))*G(2)*G(2)

F(6)=-.5*BQ*G(2)*G(2)/X(1)

F(7)=-{(2.*X(1)+.5*X(3))*G(2)*G(3)}/{3.*X(1)*X(8)}

F(8)=-{(.75*X(1)+.125*X(3))*G(2)*G(4)+(.5*X(1)+X(3)/12.)*G(3)*

1G(3)}/{X(1)*X(8)}

F(10)=-2.*YF*YA*X(8)*G(5)*G(2)/BA

F(12)=-YF*X(8)*(6.*YA*G(5)*G(4)+(4.*(YD-2.*X(3))*G(3)*G(6)+6.*YA

1*G(2)*G(7)-4.*(YD-2.*X(3))*G(2)*G(3))/3.)/BA+YH*(2.*X(1)+.5*X(3))

2*G(3)*G(2)/(3.*BC)

F(13)=-.25*YF*{(3.*X(1)-X(3))*(24.*G(4)*G(6)+16.*G(3)*G(7))+8.*

$$1YA * G(2) * G(8) - YE * (4. * G(3) * G(3) + 6. * G(2) * G(4)) / BA + YH * (3. * X(1) + .5 *$$

$$2X(3)) * (.25 * G(2) * G(4) + G(3) * G(3) / 6.) / BC$$

$$F(14) = X(4) * X(8) * (YA * (144. * G(5) * G(4) + 48. * G(2) * G(7)) + 32. * (8. * X(1)$$

$$1 - 3. * X(3)) * G(3) * G(6) - (22. * X(1) - 9. * X(3)) * G(2) * G(3)) / X(7) - X(8) * YA * ($$

$$24. * G(5) * G(3) + 2. * G(2) * G(6) - 1.5 * G(2) * G(2)) - 4. * YH * (4. * X(1) + X(3)) *$$

$$3G(2) * G(3) + .5 * BA * BQ * G(2) * G(2) / X(1)$$

$$F(15) = 24. * X(8) * X(4) * ((3. * X(1) - X(3)) * (12. * G(4) * G(6) + 8. * G(3) * G(7)) +$$

$$14. * YA * G(2) * G(8) - 6. * YE * G(3) * G(3) - 12. * (YD - 2. * X(3)) * G(2) * G(4)) / X(7) -$$

$$26. * YA * X(8) * G(5) * G(4) - X(8) * (4. * (YD - 2. * X(3)) * (G(3) * G(6) - G(2) * G(3)) +$$

$$36. * YA * G(2) * G(7)) / 3. - YH * (6. * X(1) + X(3)) * (4. * G(3) * G(3) + 6. * G(2) * G(4))$$

$$4 + BA * (4. * X(1) + X(3)) * G(2) * G(3) / (6. * X(8) * X(1))$$

$$F(16) = -3. * (8. * X(1) + X(3)) * G(3) * G(4) / (20. * X(8) * X(1))$$

$$F(17) = -3. * (10. * X(1) + X(3)) * G(4) * G(4) / (40. * X(8) * X(1))$$

$$F(18) = -X(4) * X(8) * (6. * (YD + YE - X(3)) * G(4) * G(7) + 4. * (YE + YC - X(3)) * G(3) +$$

$$1 * G(8) - 36. * YB * G(3) * G(4)) / (5. * X(7) * BC) + .3 * YH * (4. * X(1) + .5 * X(3)) * G(3)$$

$$2 * G(4) / BC$$

$$F(19) = -X(4) * X(8) * (12. * YB * G(4) * G(8) - 3. * (YC + YD - X(3)) * G(4) * G(4))$$

$$1 / (2. * X(7) * BC) + .3 * YH * (2.5 * X(1) + .25 * X(3)) * G(4) * G(4) / BC$$

$$DO 50 K=1, N$$

$$SUM=0$$

$$DO 41 L=1, N$$

$$41 SUM=SUM+C(K, L) * F(L)$$

```

50 H(K)=SUM
    WRITE OUTPUT TAPE 6,5,N,(H(I),I=1,N)
5  FORMAT(///44X,6HSECONDIX,3HSETIX,12HCUEFFICIENTS//10X,
          I5/(20X,4E20
          1.8)//)
    WD=0
    DO 60 I=1,10
    WD=WD+0.2
    FACT=1./128.
    Q=FACT*(G(1)+H(1)*WD)*WD
60 WRITE OUTPUT TAPE 6,6,WD,Q
6  FORMAT(30X,2E25.8)
    GO TO 1
END
SUBROUTINE INVERT (N,A,C)
    DIMENSION A(25,50), B(25,50), C(25,25)
    K = N
    KF = N + 1
    KK = N + N
    DO 30 I = 1, N
    K = K + 1
        DO 31 J = KF, KK
31      A(I, J) = 0.
30      A(I, K) = 1.
        DO 32 I = 1, N
32      B(I, 1) = A(I,1)
        R = 1.0/A(1,1)
        DO 33 I = 2, KK

```

```

33 B(1,I) = R * A(1, I)
      DO 34 I = 2, N
          DO 35 J = I,N
              B(J, I) = A(J, I)
              LAST = I- 1
              DO 36 J1 = 1, LAST
36          B(J,I) = B(J,I) - B(J,J1)*B(J1,I)
35          CONTINUE
              KOL = I + 1
              DO 37 J = KOL, KK
                  B(I, J) = A(I, J)
                  DO 38 J1 = 1, LAST
38          B(I,J) = B(I,J) - B(J1, J)*B(I,J1)
                  B(I,J) =B(I,J)/B(I,I)
37          CONTINUE
34 CONTINUE
      K1 = N+ 1
      NT = N - 1
      DO 39 I = 1,N
          NK = N + I
          C(N, I) = B(N,NK)
              DO 40 J = 1,NT
                  NJ = N - J
                  C(NJ, I) = B(NJ,NK)
                      DO 41 J3 = 1, J
                          NJL = K1 -J3
41          C(NJ, I) = C(NJ, I) - C(NJL, I)*B(NJ, NJL)

```

40 CONTINUE

39 CONTINUE

RETURN

END

*	DATA									
	9	52.7	52.7	36.2	1.35	1.35	.599	931.	2.	.3
	9	52.7	52.7	36.2	1.35	1.35	.599	931.	1.5	.3
	9	52.7	52.7	36.2	1.35	1.35	.599	931.	1.	.3
	9	52.7	52.7	36.2	1.35	1.35	.599	931.	.5	.3

APPENDIX C

COMPUTER PROGRAM 1
SHALLOW SPHERICAL SHELL

DIMENSION A(25,50), B(25,50), C(25,25), F(25), G(15),
H(25),X(10)

1 READ INPUT TAPE 5,2,M,(X(I), I=1,M)

2 FORMAT (1110, 8F8.3)

WRITE OUTPUT TAPE 6,3,M, (X(I), I=1,M)

3 FORMAT (///40X,5H DATA//40X, 14/(36X,E14.4))

BA=(2.*X(3)-4.*X(1))*X(8)*X(8)

BB=2.*(X(1)+X(2)-X(3))*X(8)*X(8)

BC=X(4)+X(6)

BD=X(5)+X(6)

BE=X(4)/X(7)-X(6)*BC/X(7)

BF=BA*X(3)*X(4)/(X(1)*X(7))

BQ=(0.5*X(1)+.25*X(3))/X(8)

YA=2.*X(1)-X(3)

YB=4.*X(1)-X(3)

YC=YB+2.*X(1)

YD=YC+2.*X(1)

YE=10.*X(1)-3.*X(3)

C SET UP FIRST MAIRIX

N=7

DO 10 I=1,N

DO 10 J=1,N

10 A(I,J)=0

```

A(1,1)=.5
A(1,3)=-32.
A(1,7)=.5*BA
A(2,2)=1.
A(2,3)=1.
A(3,2)=1.
A(3,3)=2.
A(4,4)=1.
A(4,5)=1.
A(4,6)=1.
A(5,5)=1.
A(5,2)=-(.5+.25*X(3)/X(1))
A(6,6)=1.
A(6,3)=-(2./3.+X(3)/(6.*X(1)))
A(7,7)=-.5
A(7,4)=1.
CALL INVERT (N,A,C)
DO 15 I=1,N
15 F(I)=0
F(1)=-.5*BA*X(3)/X(1)
F(2)=-1.
F(7)=.5*X(3)/X(1)
DO 20 I=1,N
SUMA=0.
DO 19 J=1,N
19 SUMA=SUMA+C(I,J)*F(J)
20 G(I)=SUMA

```

WRITE OUTPUT TAPE 6,4,N,(G(I), I=1,N)

4 FORMAT (///45X,5HFIRST1X,3HSET1X,12HCOEFFICIENTS//10X,
15/(20X,4E20

1.8))

G(9)=G(7)

G(8)=0

G(7)=G(6)

G(6)=G(5)

G(5)=G(4)

G(4)=0

G(10)=0

G(11)=0

C SET UP SECOND MATRIX

N=13

DO 35 I=1,N

DO 35 J=1,N

35 A(I,J)=0

A(1,1)=0.5

A(1,2)=-2.*BF

A(1,3)=-32.*(1.+BC)

A(1,4)=1152.*BE

A(1,9)=.5*BA

A(5,5)=1.

A(5,6)=1.

A(5,7)=1.

A(5,8)=1.

A(6,6)=1.

A(6,2)=-{0.5+0.25*X(3)/X(1)}

$$A(7,7)=1.$$

$$A(7,3)=-(2./3.+X(3)/(6.*X(1)))$$

$$A(8,8)=1.$$

$$A(8,4)=-(.75+.125*X(3)/X(1))$$

$$A(9,5)=1.$$

$$A(9,9)=-0.5$$

$$A(2,2)=.25*BA*X(3)/X(1)$$

$$A(2,4)=-144.$$

$$A(3,2)=1.$$

$$A(3,3)=1.$$

$$A(3,4)=1.$$

$$A(3,10)=1.$$

$$A(3,12)=1.$$

$$A(4,2)=1.$$

$$A(4,3)=2.$$

$$A(4,4)=3.$$

$$A(4,10)=4.$$

$$A(4,12)=5.$$

$$A(5,11)=1.$$

$$A(5,13)=1.$$

$$A(10,3)=BA*X(3)/(6.*X(1))$$

$$A(10,10)=-384.$$

$$A(11,10)=-(.8+.1*X(3)/X(1))$$

$$A(11,11)=1.$$

$$A(12,12)=-800.$$

$$A(12,4)=.125*BA*X(3)/X(1)$$

$$A(13,12)=-(5./6.+X(3)/(12.*X(1)))$$

A(13,13)=1.

CALL INVERT (N,A,C)

DO 40 I=1,N

40 F(I)=0

F(1)=4.*X(8)*X(4)*(8.*YA*G(5)*G(3)+4.*YA*G(2)*G(6)-3.*
YA*G(2)

1*G(2))/X(7)-2.*X(8)*YA*G(5)*G(2)+2.*BA*X(4)*(X(1)+.5*X
(4))*G(2)

2*G(2)/(X(1)*X(7)*X(8))

F(2)=(1.5*X(8)*YA+BA*(2.*X(1)+X(3))/(X(1)*8.*X(8)))*G(
2)*G(2)-

1X(8)*(4.*YA*G(5)*G(3)+2.*YA*G(2)*G(6))

F(6)=BQ*G(2)*G(2)/(2.*X(1))

F(6)=-F(6)

F(7)=((BQ+X(1)/X(8))*G(2)*G(3)+BQ*G(2)*G(3))/(3.*X(1))

F(7)=-F(7)

F(8)=(.25*(BQ+2.*X(1)/X(8))*G(2)*G(4)+(BQ+X(1)/X(8))*G
(3)*G(3))/3.

1+.25*BQ*G(2)*G(4))/X(1)

F(8)=-F(8)

F(10)=-6.*X(8)*YA*G(5)*G(4)+X(8)*(4.*(8.*X(1)-3.*X(3))
*G(3)*G(6)+

16.*YA*G(2)*G(7)-4.*(8.*X(1)-3.*X(3))*G(2)*G(3))/3.+BA*
(4.*X(1)

2+X(3))*G(3)*G(2)/(6.*X(1)*X(8))

F(12)=-X(8)*((12.*X(1)-4.*X(3))*G(3)*G(7)-1.5*(10.*X(1)
)-3.*X(3))

1*G(3)*G(3))+BA*(6.*X(1)+X(3))*G(3)*G(3)/(12.*X(1)*X(8)
)

DO 50 K=1,N

SUM=0

```

      DO 41 L=1,N
41  SUM=SUM+C(K,L)*F(L)
50  H(K)=SUM

      WRITE OUTPUT TAPE 6,5,N,(H(I),I=1,N)
5  FORMAT(///44X,6HSECONDIX,3HSETIX,12HCUEFFICIENTS//10X,
          15/(20X,4E20
          1.8)//)

      W0=0
      DO 60 I=1,10
      W0=W0+0.2
      FACT=1./64.
      Q=FACT*(G(1)+H(1)*W0)*W0
60  WRITE OUTPUT TAPE 6,6,W0,Q
6  FORMAT(30X,2E25.8)
      GO TO 1
      END
      SUBROUTINE INVERT (N,A,C)
      DIMENSION A(25,50), B(25,50), C(25,25)
      K = N
      KF = N + 1
      KK = N + N
      DO 30 I = 1, N
      K = K + 1
      DO 31 J = KF, KK
31      A(I, J) = 0.
30      A(I, K) = 1.
      DO 32 I = 1, N
32  B(I, 1) = A(I,1)

```

```

R = 1.0/A(1,1)

DO 33 I = 2, KK
33 B(1,I) = R * A(1, I)

DO 34 I = 2, N
    DO 35 J = 1,N
        B(J, I) = A(J, I)
        LAST = I- 1
        DO 36 J1 = 1, LAST
36            B(J,I) = B(J,I) - B(J,J1)*B(J1,I)
35        CONTINUE
        KOL = I + 1
        DO 37 J = KOL, KK
            B(I, J) = A(I, J)
            DO 38 J1 = 1, LAST
38                B(I,J) = B(I,J) - B(J1, J)*B(I,J1)
                    B(I,J) =B(I,J)/B(I,I)
37            CONTINUE
34 CONTINUE
    K1 = N+ 1
    NT = N - 1
    DO 39 I = 1,N
        NK = N + I
        C(N, I) = B(N,NK)
        DO 40 J = 1,NT
            NJ = N - J
            C(NJ, I) = B(NJ,NK)
            DO 41 J3 = 1, J

```

$$NJL = K1 - J3$$

41 $C(NJ, I) = C(NJ, I) - C(NJL, I) * B(NJ, NJL)$

40 CONTINUE

39 CONTINUE

RETURN

END

* DATA

8	24.	24.	16.4	0.	0.	0.	0.	2.
8	24.	24.	16.4	0.	0.	0.	0.	1.5
8	24.	24.	16.4	0.	0.	0.	0.	1.0
8	24.	24.	16.4	0.	0.	0.	0.	0.5

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BIOGRAPHICAL SKETCH

The author was born September 10, 1932, in Soochow, China. He graduated from the National Taiwan University in June, 1953, and received his B.S. degree in Mechanical Engineering.

After his graduation, he was drafted into the army, in which he served two years to fulfill his military obligations. In 1955, he returned to his alma mater where he served as a teaching assistant in the Department of Mechanical Engineering until 1959.

Since February, 1960 he has been enrolled in the Graduate School of the University of Florida and has worked as a research assistant in the Department of Engineering Mechanics. He received the degree of M.S. in Engineering in June, 1961 and was awarded the Graduate School Fellowship for the period of September, 1961 through June, 1962. He worked for the Babcock & Wilcox Company, Research Center, as an analytical engineer from June, 1962 until August, 1963. Since September, 1963 he has resumed the doctorate program in the Department of Engineering Science and Mechanics, University of Florida, from which he received his doctorate degree in April, 1965.

He is married to the former Mary Mai-li Chan and has two fine boys in the family. He is also a member of Phi Kappa Phi Society.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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